Write clear and complete solutions for each problem. The suggested exercises from Munkres book are optional and you do not have to turn them in.

Send your problem set to the grader (lashton@purdue.edu) and me (manuelr@purdue.edu) through email either as a .tex file or as a legible picture of your handwritten solutions.

Problem set 1- due September 12.

(1) Let X be a set. A *notion of proximity on* X is a subset $\sim \subseteq X \times \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the set of subsets of X, satisfying the following axioms:
- if $x \sim A$ (this notation means $(x, A) \in \sim$ and it is read as x *is near* A) then A is non-empty.
- if $x \in A$ then $x \sim A$
- if $x \sim (A \cup B)$ then $x \sim A$ or $x \sim B$.
- if $x \sim A$ and for all $a \in A$ we have $a \sim B$, then $x \sim B$.

Denote by $\mathcal{N}_X$ the set of all notions of proximity on X, i.e.

$$\mathcal{N}_X = \{\sim \subseteq X \times \mathcal{P}(X) | \sim \text{ satisfies the three axioms above }\}.$$

Denote by $\mathcal{T}_X$ the set of all topologies on X, i.e.

$$\mathcal{T}_X = \{\tau \subseteq \mathcal{P}(X) | \tau \text{ satisfies the axioms of a topology }\}.$$

Define a map of sets

$$\alpha_X : \mathcal{N}_X \to \mathcal{T}_X$$

by letting

$$\alpha_X(\sim) = \{U \subseteq X | \text{ for all } x \in U \text{ we have } (x, X - U) \notin \sim\}.$$

Prove α_X is a bijection.

(2) Denote by $\mathbb{R}$ the set of real numbers. An *open interval* is a subset $D \subset \mathbb{R}$ such that there exists $a, b \in \mathbb{R}$, $a < b$, and

$$D = (a, b) = \{y \in \mathbb{R} | a < y < b\}.$$

In a one-variable calculus course continuity is usually defined as follows:

Let D be an open interval. A function $f : D \to \mathbb{R}$ is said to be *ϵ-δ-continuous at* $c \in D$ if for any given real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $x \in D$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. We say f is *ϵ-δ-continuous* if it is ϵ-δ-continuous at all $c \in D$.

For any $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ we define $x \sim A$ if for all real numbers $r > 0$ there exist some $a \in A$ such that $|a - x| < r$. This clearly gives rise to a notion of proximity on $\mathbb{R}$.

If D is an open interval and $f : D \to \mathbb{R}$ a function, prove the following are equivalent:

i) f is ϵ-δ-continuous at $c \in D$

ii) for all $A \subseteq D$ such that $c \sim A$ then $f(c) \sim f(A)$

iii) for any open interval $(a, b) \subset \mathbb{R}$ containing $f(c)$, there exists some open interval $(a', b') \subset f^{-1}((a, b))$ containing c.

(3) a) Suppose X is a set and $\{\mathcal{T}_\alpha\}_{\alpha \in J}$ is a collection of topologies on X indexed by some set J. Is the intersection

$$\bigcap_{\alpha \in J} \mathcal{T}_\alpha$$

a topology on X? If yes, explain with a proof, if not give a counterexample.

b) Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X. Is the union $\mathcal{T} \cup \mathcal{T}'$ a topology on X? If yes, explain with a proof, if not give a counterexample.

(4) For this problem consider the set of complex numbers $\mathbb{C}$ equipped with the standard topology (through the identification with $\mathbb{R}^2$ by sending $a + bi$ to (a, b)) and $\mathbb{C} \times \mathbb{C}$ with the product topology. Prove that
a) the complex conjugation map $\mathbb{C} \to \mathbb{C}$, defined by sending $a + bi$ to $a - bi$, is continuous.
b) the complex multiplication map

$$\mu : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

$$\mu(a + ib, c + id) = (ac - bd) + i(ad + bc)$$

is continuous.

(5) A topological space X is said to be **connected** if the only subsets of X that are simultaneously closed and open are the empty set and X itself.
a) Prove that X is connected if and only if the only continuous functions from X to $\{0, 1\}$ (with the discrete topology) are the constant functions.
b) Suppose X and Y are topological spaces and $f : X \to Y$ a surjective continuous function. Prove that if X is connected, then Y is also connected.

(6) Let X be a set and let $\mathcal{T} = \{U \subset X | X - U \text{ is finite or U is empty}\}$.
a) Show $\mathcal{T}$ a topology. This topology is called the **finite complement topology** on X.
b) Let X be an infinite set equipped with the finite complement topology. Prove that if y is any point in X and (x_n) is any sequence of distinct points in X, then (x_n) converges to y. This implies, for example, that in the finite complement topology in $\mathbb{R}$, the sequence $1, \frac{1}{2}.\frac{1}{3}, \frac{1}{4}, ...$ converges to 10000000.
c) Put the following topology on the set of complex numbers $\mathbb{C}$: declare a set $C \subset \mathbb{C}$ to be *closed* if C is the set of common zeros of a set of polynomials in one variable. Prove that this defines a topology on $\mathbb{C}$ which agrees with the finite complement topology.
d) More generally, consider the set $\mathbb{C}^n$, the cartesian product of n-copies of the complex numbers. Declare a set $C \subset \mathbb{C}^n$ to be closed if C is the set of common zeros of a set of polynomials in n-variables. Prove this defines a topology on $\mathbb{C}^n$.
e) Prove that the topological space described in problem (d) has the property that for any two distinct points z and z' in $\mathbb{C}^n$, there exists a closed set C such that $z \in C$ but $z' \notin C$.

Exercises from Munkres: 13.1, 13.3, 13.8, 16.1, 16.3, 16.4, 16.6, 16.9, 16.10, 17.5, 17.8, 17.13, 17.19, 18.5, 18.8, 18.9, 18.13, 19.1, 19.2, 19.3, 19.4, 20.3,20.8

Problem set 2- due September 26

(1) Suppose X is a topological space and A a subspace of X. Recall the interior of A is defined as the uinon of all open sets contained in A. Suppose A is connected. Is the interior of A connected as well? If yes, prove it; if not, give a counterexample and explain it.
(2) Prove that if $\{X_\alpha\}_{\alpha\in J}$ is a collection of Hausdorff spaces indexed by a set J, then the cartesian product $\prod_{\alpha\in J} X_\alpha$, equipped with the product topology, is Hausdorff as well.
(3) Prove there is a homeomorphism $\mathbb{R} \to (0,1)$.
(4) Let $S^1 = \{x \times y \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ be the circle with subspace topology.
a) Prove S^1 is not homeomorphic to an open interval.
b) Prove S^1 is not homeomorphic to a closed interval.
(5) Suppose X is an infinite set equipped with the cofinite topology (open sets are the empty set and all subsets with a finite complement). Show that X is not metrizable.
(6) Let X be a topological space and $A \subset X$ a subset. A a continuous map $r : X \to A$ is called a *retraction* if $r(a) = a$ for all $a \in A$. Prove that a retraction is a quotient map.
(7) Prove that if Y is connected, $q : X \to Y$ is a quotient map, and $q^{-1}(\{y\})$ is connected for all $y \in Y$, then X is connected. Give a counter-example for this statement when Y is not connected.

Other exercises from Munkres book which will not be graded but you should try: all of the exercises in section 20.1, 20.3, 20.6, 20.9, 21.2, 21.6, all of the exercises in section 22, 23.2, 23.3, 23.6, 23.12, 24.1, 24.4, 26.6, 26.11, 27.6, Read section 28 about limit point compactness.

Problem Set 3- due October 10

(1) Let X be a compact Hausdorff space.
a) Prove that for any two disjoint closed sets C and D in X there are disjoint open sets U and V in X such that $C \subset U$ and $D \subset V$.
b) Prove that given any point $x \in X$ and an open set U containing x, there is always another open set $V \subset U$ such that $x \in V$ and $\overline{V} \subset U$, where $\overline{V}$ denotes the closed of V in X.
(2) Is the set of real numbers equipped with the cofinite topology (open sets are the empty set and all subsets with a finite complement) a compact space? Explain your answer.
(3) Let $f : X \to Y$ be a continuous function where X is compact and Y is Hausdorff. Prove f is a closed map, i.e. f carries closed sets to closed sets.
(4) Show that if Y is compact, then the projection map to the first coordinate $\pi_1 : X \times Y \to X$ is a closed map.
(5) Let X be a metric space with metric d. For subsets $C, D \subset X$ define

$$d(C, D) = \inf\{\, d(a, b) \mid a \in C \text{ and } b \in D \,\}.$$

Show that if C and D are compact then there exist points $a \in C$ and $b \in D$ with $d(a, b) = d(C, D)$, so in particular $d(C, D) > 0$ if $C \cap D = \emptyset$. Show by an example that both these statements can be false if C and D are only assumed to be closed instead of compact.

(6) Prove that for any $x \in \mathbb{R}$, the space $\mathbb{R} - \{x\}$ is not path-connected.
(7) Prove that if $\{X_\alpha\}_{\alpha \in J}$ is a collection of path-connected spaces then the product $\prod_{\alpha \in J} X_\alpha$ is path-connected as well.
(8) Let $\mathbb{Z}$ and $\mathbb{Q}$ be the subsets of integer and rational numbers, respectively, of $\mathbb{R}$ considered as topological spaces with the subspace topology. Is there a continuous bijection between $\mathbb{Z}$ and $\mathbb{Q}$.
(9) Suppose X is a Hausdorff topological spaces and

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

a nested sequence of non-empty compact sets. Prove that the intersection $\bigcap_{i=1}^{\infty} C_i$ is non-empty.

Other exercises from Munkres book which will not be graded but you should try: 26.6, 26.11, 27.6, Read section 28 about limit point compactness.

Problem set 4- due October 24

(1) Given two points x and y in a topological space X, a collection $A_1, \dots, A_n$ of subsets of X is called a *necklace from x to y* if $x \in A_1$, $y \in A_n$, and $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Prove that if x and y are two points of a connected space X, and $\mathcal{A}$ an open cover of X, then there is a necklace from x to y consisting of open sets $A_1, \dots, A_n$ with $A_i \in \mathcal{A}$ for $i = 1, \dots, n$.
(2) Let X is a locally compact Hausdorff space, Z is a Hausdorff space, and Y any topological space. Prove that the map

$$\alpha \colon \mathcal{C}(X \times Z, Y) \to \mathcal{C}(Z, \mathcal{C}(X, Y))$$

defined on any $f \colon X \times Z \to Y$ by

$$\alpha(f)(z)(x) = f(x, z),$$

for any $x \in X$, $z \in Z$, is a homeomorphism.
(3) Let X be a locally compact Hausdorff space and let $Z \subset X$ be a subset with the property that $Z \cap K$ is closed for every compact $K \subset X$. Prove that Z is closed.
(4) Let d be the standard metric in the Euclidean plane $\mathbb{R}^2$ and denote by $\mathbf{0} = (0, 0) \in \mathbb{R}^2$. Define a new map

$$d' : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

by

$$d'(p, q) = d(\mathbf{0}, p) + d(\mathbf{0}, q)$$

for any $p \neq q$; and if $p = q$, we let $d'(p, q) = 0$.
 (a) Prove d' defines a metric on $\mathbb{R}^2$.
 (b) Prove that, under the metric d', if $p \neq 0$, then $\{p\}$ is an open set.
 (c) Prove the metric space $(\mathbb{R}^2, d')$ is not compact.
 (d) Is $(\mathbb{R}^2, d')$ locally compact?
 (e) Is the subspace $\mathbb{R}^2 - \{\mathbf{0}\} \subset (\mathbb{R}^2, d')$ locally compact?
(5) For n a positive integer and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let

$$\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}.$$

Let k be a positive integer and denote

$$S^k := \{x \in \mathbb{R}^{k+1} : \|x\| = 1\} \quad \text{and} \quad D^k := \{y \in \mathbb{R}^k : \|y\| \leq 1\}.$$

Let $\sim_1$ be the equivalence relation on S^k generated by the relation

$$(x_1, \dots, x_k, x_{k+1}) \sim_1 (x_1, \dots, x_k, -x_{k+1}) \quad \text{for } (x_1, \dots, x_{k+1}) \text{ such that } x_{k+1} \neq 0.$$

Show that the quotient space $S^k/\sim_1$ is homeomorphic to D^k.

(6) Let k be a positive integer. Let $\sim_2$ be the equivalence relation on $\mathbb{R}^{k+1} - \{0\}$ given by

$$x \sim_2 tx \quad \text{for } x \in \mathbb{R}^{k+1} - \{0\} \text{ and } t \in \mathbb{R} - \{0\}.$$

The quotient space $(\mathbb{R}^{k+1} - \{0\})/\sim_2$ is denoted $\mathbb{RP}^k$ and called the *k-dimensional real projective space*. This is a space whose points may be identified with lines in $\mathbb{R}^{k+1}$ passing through the origin. The equivalence class of $(x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} - \{0\}$ with respect to $\sim_2$ is denoted by

$$[x_0 : x_1 : \cdots : x_k] \in \mathbb{RP}^k.$$

(a) Show that there is a bijection between $\mathbb{RP}^k$ and the set of lines in $\mathbb{R}^{k+1}$ that pass through the origin.

(b) Show that $\mathbb{RP}^k$ is locally Euclidean of dimension k.
Hint: For each $0 \leq i \leq k$, consider the subspace of $\mathbb{RP}^k$ consisting of equivalence classes $[x_0 : x_1 : \cdots : x_k]$ such that $x_i \neq 0$.

(c) Let S^k and D^k be as in Problem 3. Let $\sim_3$ be the equivalence relation on S^k generated by the relation $x \sim_3 -x$ for $x \in S^k$, and let $\sim_4$ be the equivalence relation on D^k generated by the relation $y \sim_4 -y$ for $y \in S^{k-1}$. Show that the quotient spaces $S^k/\sim_3$ and $D^k/\sim_4$ are both homeomorphic to $\mathbb{RP}^k$.

(d) Let $\sim_5$ be the equivalence relation on $[0,1] \times [0,1]$ generated by the relations

$$(t, 0) \sim_5 (1-t, 1) \quad \text{and} \quad (0, t) \sim_5 (1, 1-t) \quad \text{for } t \in [0,1].$$

Show that the quotient space $([0,1] \times [0,1])/\sim_5$ is homeomorphic to $\mathbb{RP}^2$.

Problem set 5

Read and study the proof of the classification of surfaces theorem.

Problem Set 6 — due November 30

(1) Let X be a space and $x_0 \in X$ a point. The underlying set of the fundamental group $\pi_1(X, x_0)$ can be understood as the set of basepoint-preserving homotopy classes of maps

$$(S^1, b) \to (X, x_0)$$

from the circle S^1 with basepoint $b = (1, 0)$ into (X, x_0).

Consider the set $[S^1, X]$ of homotopy classes of maps $S^1 \to X$ with *no* restriction on basepoints. There is a natural map

$$\psi : \pi_1(X, x_0) \to [S^1, X].$$

Prove that if X is path-connected then:

(a) ψ is surjective.

(b) For loops α and β in X based at x_0,

$$\psi([\alpha]) = \psi([\beta]) \quad \text{if and only if} \quad [\alpha] \text{ and } [\beta] \text{ are conjugate in } \pi_1(X, x_0),$$

i.e. there exists $[\gamma] \in \pi_1(X, x_0)$ such that

$$[\alpha] = [\gamma]^{-1} * [\beta] * [\gamma].$$

(2) Prove that for any space X the following conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0)$ is the trivial group for all $x_0 \in X$.

(3) Prove that for any homomorphism of groups

$$\phi : \pi_1(S^1, b) \to \pi_1(S^1, b)$$

there exists a continuous map

$$f : (S^1, b) \to (S^1, b)$$

such that $f_* = \phi$.

(4) Let X be a space, $A \subset X$ a subspace, and $x_0 \in A$.

(a) Give an example of X and A such that the inclusion $i : A \hookrightarrow X$ induces a non-injective map $\pi_1(A, x_0) \to \pi_1(X, x_0)$.

(b) A *retraction* is a map $r : X \to A$ whose restriction $r|_A$ is the identity. Prove that if there exists a retraction $r : X \to A$, then the induced map $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective.

(c) Prove there are no retractions $r : X \to A$ when:

(i) $X = \mathbb{R}^3$ and A is any subspace homeomorphic to S^1.

(ii) $X = S^1 \times D^2$ (the solid torus) and $A = S^1 \times S^1$ (its boundary).

(5) Suppose E is path-connected and B is simply connected. Prove that any covering map $p : E \to B$ is a homeomorphism.

(6) (a) Let p_1 and p_2 be two distinct points in $\mathbb{R}^3$. Show that $\mathbb{R}^3 - \{p_1, p_2\}$ is simply connected.

(b) Let $p_1, \ldots, p_n$ be n distinct points on S^2, and let $X = S^2 - \{p_1, \ldots, p_n\}$. Compute $\pi_1(X, x)$ for a basepoint $x \in X$.

(7) Let $X = \mathbb{R}^3$ with the z-axis removed, and let $x_0 = (1, 0, 0)$. Determine $\pi_1(X, x_0)$ and prove your answer.
(8) Compute the fundamental group of the space obtained by removing two distinct points from the torus $S^1 \times S^1$.
(9) Let $h : X \to Y$ be a continuous map that induces the trivial homomorphism of fundamental groups. Let $x_0, x_1 \in X$, and let γ and γ' be paths from x_0 to x_1. Prove that $h \circ \gamma$ and $h \circ \gamma'$ are path homotopic.

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Problem Set 7 — do not turn in, consider it as practice for the oral final

(1) The following problem is taken from the previous homework for which we had not discussed Van Kampen's theorem yet.
  (a) Let p_1 and p_2 be two distinct points in $\mathbb{R}^3$. Show that $\mathbb{R}^3 - \{p_1, p_2\}$ is simply connected.
  (b) Let $p_1, \ldots, p_n$ be n distinct points on S^2, and let $X = S^2 - \{p_1, \ldots, p_n\}$. Compute $\pi_1(X, x)$ for a basepoint $x \in X$.
(2) Let X be any path-connected space. Suppose A is another space homotopy equivalent to a point. Prove the fundamental groups of X and $X \vee A$ are isomorphic.
(3) Suppose $(X_1, b_1), \ldots, (X_n, b_n)$ are n pointed path-connected topological spaces such that for each $i = 1, ..., n$ there is a open set $N_i \subset X_i$ such that $b_i \in N_i$ and the inclusion $\{b_i\} \to N_i$ is a homotopy equivalence. Prove that the fundamental group of the wedge $X_1 \vee ... \vee X_n$ is isomorphic to the free product of the groups $\pi_1(X_1, x_1), ..., \pi_1(X_n, x_n)$.
(4) A *compact orientable surface of genus g* can be defined as the quotient space of a $4g$-gon (a polygon with $4g$ sides) by cyclically labeling its edges as

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, a_g^{-1}$$

and then making identifications according to the corresponding orientations (for example, edges a_i and a_i^{-1} have opposite orientations). Use Van Kampen's Theorem to prove that the fundamental group of a surface of genus g can be presented as the group generated by $2g$ elements

$$a_1, b_1, \ldots, a_g, b_g$$

with a single relation given by

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

(5) Prove that for any group G there is a path-connected space X with fundamental group isomorphic to G. Hint: Use the fact that any group has a presentation to construct the desired space by starting with a wedge of circles labeled by the generators and gluing 2-dimensional disks according to the relations.