COMPARING DG CATEGORY MODELS FOR PATH SPACES VIA A_∞-FUNCTORS

MANUEL RIVERA AND YI WANG

Abstract. We construct a many-object dual version of Chen's iterated integral map. For any topological space X, the construction takes the form of an A_∞-functor between two dg categories whose objects are the points of X: the domain has as morphisms the singular (cubical) chains on the space of (Moore) paths in X and the codomain has morphisms arising by totalizing a cosimplicial chain complex determined by the dg coalgebra of singular (simplicial) chains in X. When X is simply connected, we show this construction defines a homotopy inverse to a classical map of Adams, which sends ordered sequences of singular simplices in X linked by shared vertices to cubes of paths in X. When X is not necessarily simply connected, following an idea of Irie, we incorporate the fundamental groupoid of X into the construction and deduce analogous results. Along the way, we provide an elementary and new proof of the fact that the (direct-sum) cobar construction of the chains in X, suitably interpreted, models the dg category of paths in X, an extension of Adams's cobar theorem established by Rivera-Zeinalian using different methods.

1. Introduction

Denote by

$$\mathsf{P}\colon \mathsf{Top} \to \mathsf{Cat}_{\mathsf{Top}}$$

the functor that associates to any topological space[1] X the topologically enriched category $\mathsf{P}X$ defined as follows. The objects of $\mathsf{P}X$ are the points of X and, for any two $a, b \in X$, the morphism space $\mathsf{P}X(a,b)$ is the set of all pairs (γ, T), where $T \in \mathbb{R}_{\geq 0}$ and $\gamma\colon [0,T] \to X$ is a continuous path with $\gamma(0) = a$ and $\gamma(T) = b$, equipped with the compact-open topology. The composition rule on $\mathsf{P}X$ is defined by concatenating paths and adding the corresponding parameters. The identity morphism at an object $b \in X$ is given by the constant path $\gamma\colon \{0\} = [0,0] \to X$, $\gamma(0) = b$. The topological monoid $\mathsf{P}X(b,b)$ is the space $\Omega_b X$ of Moore loops in X based at b.

For a fixed commutative ring with unit R, denote by

$$\mathsf{C}^{\square}\colon \mathsf{Cat}_{\mathsf{Top}} \to \mathsf{dgCat}_R$$

the functor that associates to any topologically enriched category the differential graded (dg) R-category obtained by applying the normalized cubical singular chains with coefficients in R at the level of spaces of morphisms. Similarly, denote by C^{Δ} the functor that applies normalized simplicial singular chains instead. The main purpose of this article is to compare the functor

$$\mathsf{C}^{\square} \circ \mathsf{P}\colon \mathsf{Top} \to \mathsf{dgCat}_R \tag{1.1}$$

[1] Throughout this paper, topological spaces are always assumed to be locally path-connected and semilocally simply connected.

and the following different versions of the *cobar construction*.

(1) *The classical cobar construction.* This is a functor

$$\mathsf{Cobar}\colon \mathsf{dgCoalg}_R \to \mathsf{dgAlg}_R$$

which takes a dg coaugmented (coassociative) R-coalgebra C as input and produces a dg augmented (associative) R-algebra $\mathsf{Cobar}(C)$. The underlying graded algebra of $\mathsf{Cobar}(C)$ is the free algebra $T(s^{-1}\overline{C}) = \bigoplus_{n=0}^{\infty}(s^{-1}\overline{C})^{\otimes n}$ where $\overline{C}$ is the cokernel of the coaugmentation of C and s^{-1} denotes the shift functor, i.e., $(s^{-1}\overline{C})_i = \overline{C}_{i+1}$. The differential of $\mathsf{Cobar}(C)$ is induced by the sum $\partial + \Delta$, where $\partial\colon C \to C$ is the differential of C and $\Delta\colon C \to C \otimes C$ is the coassociative coproduct of C. The functor Cobar is a left adjoint with right adjoint being the classical bar construction Bar. The main example we apply this construction to is the dg coaugmented coalgebra of singular chains on a pointed space (X, b).

(2) (See [Riv24; HL22]). *A "many object" version of the cobar construction.* This is a functor

$$\mathsf{Cobar}^{\boxtimes}\colon \mathsf{cCoalg}_R \to \mathsf{dgCat}_R$$

that takes a *categorical R-coalgebra* $\mathcal{C}$ to a dg category $\mathsf{Cobar}^{\boxtimes}(\mathcal{C})$. Three features of a categorical coalgebra $\mathcal{C}$ are (i) $\mathcal{C}$ is a non-negatively graded R-coalgebra such that $\mathcal{C}_0 = R[\mathcal{S}]$ for some set $\mathcal{S}$, (ii) the "differential" ∂ on $\mathcal{C}$ does not square to zero but its failure is controlled by a "curvature" term, and (iii) $\mathcal{C}$ has a compatible $\mathcal{C}_0$-bicomodule structure. The set of objects in $\mathsf{Cobar}^{\boxtimes}(\mathcal{C})$ is $\mathcal{S}$. For $a, b \in \mathcal{S}$, the definition of $\mathsf{Cobar}^{\boxtimes}(\mathcal{C})(a, b)$ is recalled in section 2.1; here we just point that the generators are ordered sequences of elements of $\mathcal{C}$ "connecting" a and b (expressed as monomials over the cotensor product $\boxtimes$ over $\mathcal{C}_0$) and the differential on $\mathsf{Cobar}^{\boxtimes}(\mathcal{C})(a, b)$ is induced by $\partial + \Delta + h$ where h is the curvature of $\mathcal{C}$. A natural example of a categorical coalgebra may be obtained for any topological space X with $\mathcal{S}$ being the underlying set of X by slightly modifying the dg coalgebra of singular chains on X giving a functor

$$\mathcal{C}^{\blacktriangle}\colon \mathsf{Top} \to \mathsf{cCoalg}_R.$$

For simplicity, we write $\mathcal{C}^{\blacktriangle}(X)$ as $\mathcal{C}(X)$.

(3) *A cosimplicial totalized "many-object" version of the cobar construction.* This is a functor

$$\mathsf{Cobar}^{\prod}\colon \mathsf{dgCoalg}_R^{\mathrm{obj}} \to \mathsf{dgCat}_R$$

where the input is a dg R-coalgebra C such that $C_0 = R[S]$ for some set of cycles S, which we call "objects". The set of objects in $\mathsf{Cobar}^{\prod}(C)$ is S, and for $a, b \in S$, $\mathsf{Cobar}^{\prod}(C)(a, b)$ is defined as the totalization of certain cosimplicial chain complex defined by the coalgebra structure of C (and hence the direct product symbol in the notation); see Section 2.1 for more details. An example of such a C is obtained from a topological space X with S being the underlying set of X and C being the dg coalgebra of singular chains on X; this gives a functor

$$C^{\blacktriangle}\colon \mathsf{Top} \to \mathsf{dgCoalg}_R^{\mathrm{obj}},$$

and, for simplicity, we may write $C^{\blacktriangle}(X)$ as $C(X)$.

The "many object" versions (2)(3) of the cobar construction lead to functors

$$\mathsf{Cobar}^{\boxtimes} \circ \mathcal{C}^{\blacktriangle}\colon \mathsf{Top} \to \mathsf{dgCat}_R, \tag{1.2}$$

$$\mathsf{Cobar}^{\prod} \circ C^{\blacktriangle} \colon \mathsf{Top} \to \mathsf{dgCat}_R. \tag{1.3}$$

In this article, we establish an explicit relationship between the functors (1.1), (1.2), and (1.3).

First, a classical construction of Adams' gives a natural transformation

$$\mathcal{A} \colon \mathsf{Cobar}^{\boxtimes} \circ \mathcal{C}^{\blacktriangle} \Rightarrow \mathsf{C}^{\square} \circ \mathsf{P}.$$

The idea behind the construction of $\mathcal{A}$ is to associate to any ordered sequence of simplices in X linked by a shared vertex (a "necklace" in X) a suitable cube of paths connecting the first and last vertices in the sequence.

Remark 1.1. Adams [Ada56] originally worked with $\mathsf{Cobar}(C^1(X,b))$, where $b \in X$ is a base point and $C^1(X,b)$ is the dg coaugmented coalgebra of 1-reduced singular chains on (X,b). This construction naturally extends to $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^{\blacktriangle}(X))$.

Next, for any topological space X, a variation (or "pre-dual" version) of Chen's iterated integral map gives natural chain maps

$$It_X \colon \mathsf{C}^{\square}(\mathsf{P}X)(a,b) \to \mathsf{Cobar}^{\prod}(C(X))(a,b), \tag{1.4}$$

for all $a, b \in X$, induced by evaluating each path in $\mathsf{P}X(a,b)$ at all ordered sequences of n marked points (which are determined by points in the n-simplex), for all $n \geq 0$; see Section 2.2.2 for more details.

The maps 1.4 all together do not define a (strict) functor of dg categories $\mathsf{C}^{\square}(\mathsf{P}X) \to \mathsf{Cobar}^{\prod}(C(X))$ since compositions are not preserved. However, we have the following statement, which is the first main result of this article.

Theorem 1.2. The natural maps in (1.4) can be extended to a natural A_∞-transformation

$$\mathcal{I} = \{\mathcal{I}_n\}_{n \geq 1} \colon \mathsf{C}^{\square} \circ \mathsf{P} \Rightarrow \mathsf{Cobar}^{\prod} \circ C^{\blacktriangle},$$

in the sense that for any topological space X,

$$\mathcal{I}_X = \{\mathcal{I}_{X,n}\}_{n \geq 1} \colon \mathsf{C}^{\square}(\mathsf{P}X) \to \mathsf{Cobar}^{\prod}(C(X))$$

is an A_∞-functor with $\mathcal{I}_{X,1} = It_X$, and the family $\{\mathcal{I}_X\}_{X \in \mathsf{Top}}$ is natural in X.

Remark 1.3. Chen [Che73] originally worked with cochains (differential forms) on what he called differentiable spaces, which generalize smooth manifolds. In contrast, we work with singular chains on topological spaces, leading to the map $\mathcal{I}_1$ which is formally dual to Chen's. The higher components $\mathcal{I}_n$ $(n > 1)$ can be viewed geometrically as arising from subdivisions of simplices compatible with path concatenation. These higher homotopies disappear after pairing with differential forms via integration and are therefore not seen in Chen's framework (see Remark 2.4 and the subsequent discussion for details).

A natural transformation between two functors $\mathsf{Top} \to \mathsf{dgCat}_R$ is a natural A_∞-transformation with vanishing higher components. Hence, given the natural A_∞-transformations $\mathcal{A}$ and $\mathcal{I}$, their composition $\mathcal{I} \circ \mathcal{A}$ is a natural A_∞-transformation

$$\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 1} = \{\mathcal{I}_n \circ \mathcal{A}\}_{n \geq 1} \colon \mathsf{Cobar}^{\boxtimes} \circ \mathcal{C}^{\blacktriangle} \Rightarrow \mathsf{Cobar}^{\prod} \circ C^{\blacktriangle}.$$

On the other hand, there is a straightforwardly defined natural transformation

$$\mathcal{G} \colon \mathsf{Cobar}^{\boxtimes} \circ \mathcal{C}^{\blacktriangle} \Rightarrow \mathsf{Cobar}^{\prod} \circ C^{\blacktriangle}$$

that, roughly speaking, is induced by the inclusion of a direct sum into a direct product (modulo the subtle difference between $\mathcal{C}^{\mathbb{A}}$ and $C^{\mathbb{A}}$). The following is the second main result of the present article.

Theorem 1.4. The natural A_∞-transformations $\mathcal{F}$ and $\mathcal{G}$ are A_∞-homotopic, i.e., for any topological space X, there exists a natural A_∞-homotopy $\mathcal{H}_X$ between the A_∞-functors $\mathcal{F}_X$ and $\mathcal{G}_X$.

Remark 1.5. Suppose X is simply connected, and fix a basepoint $b \in X$. If one restricts to the dg coalgebra of 1-reduced singular chains $C^1(X, b)$, then $\mathcal{G}_X$ identifies $\mathsf{Cobar}^{\boxtimes}$ with the conormalization of $\mathsf{Cobar}^{\prod}$. Thus, Theorem 1.4 implies:

> *Chen's iterated integral map, suitably extended and reinterpreted, is a left A_∞-homotopy inverse of Adams' map for simply connected topological spaces.*

See Section 3.3 for a more detailed statement.

Recall that a functor $\mathsf{C}_1 \to \mathsf{C}_2$ between dg categories is called a *quasi-equivalence* if it induces quasi-isomorphisms on all morphism complexes and an equivalence $H_0(\mathsf{C}_1) \simeq H_0(\mathsf{C}_2)$. The same notion applies to A_∞-categories and A_∞-functors, where it is called an *A_∞-quasi-equivalence*. Building on Remark 1.5, we then obtain:

Corollary 1.6. For any simply connected topological space X, the functor

$$\mathcal{A}_X \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X)) \to \mathsf{C}^{\square}(\mathsf{P}X)$$

induced by Adams' map is a quasi-equivalence if and only if the A_∞-functor

$$\mathcal{I}_X \colon \mathsf{C}^{\square}(\mathsf{P}X) \to \mathsf{Cobar}^{\prod}(C(X))$$

motivated by Chen's iterated integral map is an A_∞-quasi-equivalence.

Recall that Adams [Ada56] constructed the map $\mathsf{Cobar}(C^1(X, b)) \to C^{\square}(\Omega_b X)$ and proved it is a quasi-isomorphism for simply connected X by comparing the Serre spectral sequences associated with the path fibration. Chen [Che73] proved that his original iterated integral map, which takes an ordered sequence of differential forms on a manifold M as input and produces a cochain in $\Omega_b M$ as output, is a quasi-isomorphism for simply connected M by pairing with Adams' cobar construction and using spectral sequence arguments. Thus, Corollary 1.6 highlights the structural feature underlying Chen's proof and provides a conceptual explanation for why that argument works.

An adaptation of Adams' original argument yields that, for any simply connected space X, the dg functor $\mathcal{A}_X \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X)) \to \mathsf{C}^{\square}(\mathsf{P}X)$ is a quasi equivalence (in fact, we prove a stronger statement in this article, see Theorem 1.8 below). Together with Corollary 1.6, this immediately implies the following.

Corollary 1.7. If X is simply connected, the A_∞-functor

$$\mathcal{I}_X \colon \mathsf{C}^{\square}(\mathsf{P}X) \to \mathsf{Cobar}^{\prod}(C(X))$$

is an A_∞-quasi-equivalence.

A natural question is whether Corollaries 1.6 and 1.7 extend to possibly non-simply connected spaces. One may begin to address this by examining whether Adams' map and Chen's iterated integral map remain quasi-isomorphisms when X is not simply connected. In recent years, Adams' theorem has been shown to extend to arbitrary topological spaces [RZ18]. In fact, it turns out that $\mathcal{A}_X$ is a quasi-equivalence even when X is a non-simply connected space. We give an elementary proof (independent of all the above results) of this extension of Adams' theorem by lifting necklaces to the universal cover and using Adams' original proof for simply connected spaces.

Theorem 1.8. For any topological space X, the functor

$$\mathcal{A}_X\colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X)) \to \mathsf{C}^{\square}(\mathsf{P}X)$$

is a quasi-equivalence of dg categories.

Corollary 1.9 ([RZ18; Riv22])**.** For any pointed topological space (X, b), Adams' map

$$\mathsf{Cobar}(C^0(X,b)) \to C^{\square}(\Omega_b X)$$

is a quasi-isomorphism of dg algebras.

In contrast, Chen's iterated integral map is not a quasi-isomorphism unless additional assumptions are imposed on the fundamental group. Similarly, the map $\mathcal{I}_X$ is generally not a quasi-equivalence for an arbitrary topological space X. To address this issue in the non-simply connected case, it is necessary to modify the functor $\mathsf{Cobar}^{\prod} \circ C^{\triangle}$ and, accordingly, the natural A_∞-transformation $\mathcal{I}$. To this end, we incorporate the fundamental groupoid into the construction, following an idea of Irie [Iri17; Wan23]. The modification of $\mathsf{Cobar}^{\prod} \circ C^{\triangle}$ defines a functor

$$\mathsf{C}^{\mathsf{P}}\colon \mathsf{Top} \to \mathsf{dgCat}_R,$$

and there are natural chain maps

$$\widetilde{It}_X\colon \mathsf{C}^{\square}(\mathsf{P}X)(a,b) \to \mathsf{C}^{\mathsf{P}}(X)(a,b), \qquad X \in \mathsf{Top}, \quad a,b \in X$$

induced by evaluation maps on path spaces. There is also a natural transformation

$$\widetilde{\mathcal{G}}\colon \mathsf{Cobar}^{\boxtimes} \circ \mathcal{C}^{\triangle} \Rightarrow \mathsf{C}^{\mathsf{P}}$$

defined analogously to $\mathcal{G}$. If X is simply connected, then $\mathsf{C}^{\mathsf{P}}(X)$, $\widetilde{It}_X$, and $\widetilde{\mathcal{G}}_X$ coincide with $\mathsf{Cobar}^{\prod}(C(X))$, It_X, and $\mathcal{G}_X$, respectively.

We prove the following analogue of Theorem 1.2 and Theorem 1.4 as the fourth main result of this article.

Theorem 1.10. The collection of natural chain maps $\{\widetilde{It}_X\}_{X\in\mathsf{Top}}$ extends to a natural A_∞-transformation

$$\widetilde{\mathcal{I}}\colon \mathsf{C}^{\square} \circ \mathsf{P} \Rightarrow \mathsf{C}^{\mathsf{P}}.$$

Moreover, there exists an A_∞-homotopy $\widetilde{\mathcal{H}}$ between the natural A_∞-transformations

$$\widetilde{\mathcal{F}} = \widetilde{\mathcal{I}} \circ \mathcal{A} \text{ and } \widetilde{\mathcal{G}}\ :\ \mathsf{Cobar}^{\boxtimes} \circ \mathcal{C}^{\triangle} \Rightarrow \mathsf{C}^{\mathsf{P}}.$$

For any topological space X, an analogue of Corollary 1.6 holds where $\mathcal{F}_X$, $\mathcal{G}_X$, and $\mathcal{I}_X$ are replaced by $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{G}}_X$, and $\widetilde{\mathcal{I}}_X$, respectively. Together with Theorem 1.8, this shows that $\widetilde{\mathcal{I}}_X$ is a quasi-equivalence for any topological space X. In this sense, $\widetilde{\mathcal{I}}_X$ provides a formal and "correct" extension of Chen's iterated integral map, guaranteeing its quasi-equivalence property beyond the simply connected case. See Section 5 for further details.

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2. Main Constructions

In this section, we give further details on the cobar constructions $\mathsf{Cobar}^{\boxtimes}$, Cobar^{Π}, the natural transformations $\mathcal{A}$, $\mathcal{G}$ and the natural maps $\{It_X\}_{X \in \mathsf{Top}}$ introduced in Section 1.

2.1. **Different versions of the cobar construction.**
In this subsection, we give more details on the non-classical versions of the cobar construction.

2.1.1. *The "many object" version (2).* Given a set $\mathcal{S}$, denote by $R[\mathcal{S}]$ the cocommutative counital coalgebra given by the free R-module on $\mathcal{S}$ equipped with the coproduct determined by $b \mapsto b \otimes b$ for all $b \in \mathcal{S}$. We shall define a version of the cobar construction which takes as an input the following notion introduced in [Riv24].

A *categorical R-coalgebra* consists of a tuple $(\mathcal{C}, \partial, \Delta, h)$ where

(1) $(\mathcal{C}, \Delta)$ is a non-negatively graded coassociative counital coalgebra that is flat as an R-module
(2) $\partial : \mathcal{C} \to \mathcal{C}$ is a degree -1 coderivation of the coproduct Δ
(3) $h : \mathcal{C} \to R$ is a linear map of degree -2 satisfying $h \circ \partial = 0$ and
$$\partial^2 = (h \otimes \mathrm{id} - \mathrm{id} \otimes h) \circ \Delta,$$
i.e. h is a *curvature* for $(\mathcal{C}, \partial, \Delta)$
(4) The set
$$\mathcal{S}(\mathcal{C}) = \{x \in \mathcal{C} : \Delta(x) = x \otimes x \text{ and } \varepsilon(x) = 1\},$$
where $\varepsilon \colon \mathcal{C} \to R$ denotes the counit, is non-empty and the natural inclusion map $\mathcal{S}(\mathcal{C}) \hookrightarrow \mathcal{C}_0$ induces an isomorphism of coalgebras
$$R[\mathcal{S}(\mathcal{C})] \cong \mathcal{C}_0.$$
(5) The natural projection $\epsilon : \mathcal{C} \to \mathcal{C}_0$ satisfies $\epsilon \circ \partial = 0$

Any categorical coalgebra $(\mathcal{C}, \partial, \Delta, h)$ has a natural $\mathcal{C}_0$-bicomodule structure with structure maps given by
$$\rho_r \colon \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\mathrm{id} \otimes \epsilon} \mathcal{C} \otimes \mathcal{C}_0$$
and
$$\rho_l \colon \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\epsilon \otimes \mathrm{id}} \mathcal{C}_0 \otimes \mathcal{C}.$$

Categorical coalgebras form a category with the following notion of morphism. A morphism from $(\mathcal{C}, \partial, \Delta, h)$ to $(\mathcal{C}', \partial', \Delta', h')$ consists of a pair $f = (f_0, f_1)$ where

(1) $f_0 : (\mathcal{C}, \Delta) \to (\mathcal{C}', \Delta')$ is a morphism of graded coalgebras
(2) $f_1 : \mathcal{C} \to \mathcal{C}'_0$ is a $\mathcal{C}'_0$-bicomodule map of degree -1

satisfying

$$f_0 \circ \partial = \partial' \circ f_0 + (\overline{f}_1 \otimes f_0) \circ (\Delta - \Delta^{\mathrm{op}}) \text{ and}$$
$$h' \circ f_0 = h + \overline{f}_1 \circ \partial + (\overline{f}_1 \otimes \overline{f}_1) \circ \Delta,$$

where $\overline{f}_1 = \varepsilon' \circ f_1$ and ε' is the counit of $\mathcal{C}'$. The composition of two morphisms is defined by

$$(g_0, g_1) \circ (f_0, f_1) = (g_0 \circ f_0, g_1 \circ f_0 + g_0 \circ f_1). \tag{2.1}$$

We denote by cCoalg_R the category of categorical coalgebras.

Given a categorical coalgebra $\mathcal{C}$, the dg category $\mathsf{Cobar}^{\boxtimes}(\mathcal{C})$ is defined as follows. The objects are the elements of $\mathcal{S}$. Denote by $\boxtimes$ the cotensor product over $\mathcal{C}_0$ and by s^{-1} the degree shift functor by -1. For any two $a, b \in \mathcal{S}$ we define

$$\mathsf{Cobar}^{\boxtimes}(\mathcal{C})(a,b) = \bigoplus_{n=0}^{\infty} R_a \boxtimes (s^{-1}\overline{\mathcal{C}})^{\boxtimes n} \boxtimes R_b,$$

where, for any $a \in \mathcal{S}$, R_a is R equipped with the $R[\mathcal{S}]$-bicomodule determined by the inclusion $\{a\} \hookrightarrow \mathcal{S}$, $\overline{\mathcal{C}} = \bigoplus_{i=1}^{\infty} \mathcal{C}_i$, and $(s^{-1}\overline{\mathcal{C}})^{\boxtimes 0} = \mathcal{C}_0$. The identity morphism at $a \in \mathcal{S}$ corresponds to the unit of R through the following identification

$$1_R \in R \cong R_a \cong R_a \boxtimes (s^{-1}\overline{\mathcal{C}})^{\boxtimes 0} \boxtimes R_a \subseteq \mathsf{Cobar}^{\boxtimes}(\mathcal{C})(a,a)_0.$$

Each differential

$$D^{\boxtimes} \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C})(a,b) \to \mathsf{Cobar}^{\boxtimes}(\mathcal{C})(a,b)$$

is induced by the sum $\partial + \Delta + h$. The fact $D^{\boxtimes} \circ D^{\boxtimes} = 0$ then follows from the compatibility of ∂ and Δ, the coassociativity of Δ, and the curvature equation relating h, ∂, and Δ. The composition of morphisms is given by concatenation of monomials. This construction gives rise to a functor

$$\mathsf{Cobar}^{\boxtimes} \colon \mathsf{cCoalg}_R \to \mathsf{dgCat}_R.$$

The normalized singular chains on a topological space may be regarded as a categorical coalgebra as we now explain. Recall that the classical normalized singular chains functor

$$C_\bullet : \mathsf{Top} \to \mathsf{dgCoalg}_R$$

assigns to a space X a dg coalgebra $(C_\bullet(X), \partial, \Delta)$ with coproduct Δ given by the Alexander-Whitney diagonal approximation. This does not define a categorical coalgebra with curvature 0 since, in general, $\epsilon \circ \partial \neq 0$, where $\epsilon \colon C_\bullet(X) \to C_0(X)$ is the canonical projection map. However, one may proceed as follows. Let

$$e \colon C_\bullet(X) \to R \tag{2.2}$$

be the 1-cochain induced by sending non-degenerate singular 1-simplices to 1_R and everything else to 0. Define a degree -1 linear map

$$\widetilde{\partial} \colon C_\bullet(X) \to C_{\bullet-1}(X)$$

and a degree -2 linear map

$$h : C_\bullet(X) \to R$$

by

$$\widetilde{\partial} = \partial - (\mathrm{id} \otimes e - e \otimes \mathrm{id}) \circ \Delta$$

and

$$h = (e \otimes e) \circ \Delta + e \circ \partial,$$

respectively. A straightforward check yields that $\mathcal{C}(X) = (C_\bullet(X), \widetilde{\partial}, \Delta, h)$ defines a categorical coalgebra. Furthermore, this construction defines a functor

$$\mathcal{C}\colon \mathsf{Top} \to \mathsf{cCoalg}_R.$$

2.1.2. *The totalized cosimplicial version (3).* For any dg R-coalgebra C such that $C_0 = R[S]$, the dg category $\mathsf{Cobar}^{\prod}(C)$ is defined as follows. The set of objects is S, and for any $a, b \in S$,

$$\mathsf{Cobar}^{\prod}(C)(a,b) = \prod_{n=0}^{\infty} R_a \otimes (s^{-1}C)^{\otimes n} \otimes R_b.$$

Each differential

$$D^{\prod}\colon \mathsf{Cobar}^{\prod}(C)(a,b) \to \mathsf{Cobar}^{\prod}(C)(a,b)$$

is induced by the sum $\partial + \Delta$. The composition of morphisms is given by concatenation of monomials. The chain complex $\mathsf{Cobar}^{\prod}(C)(a,b)$ may also be described as the totalization of a cosimplicial chain complex as follows. Recall that for a cosimplicial chain complex $[n] \mapsto (C(n), \partial)$, its totalization has underlying R-module

$$\mathrm{Tot}^{\prod}_\bullet(\{C(n)\}_{n\geq 0}) = \prod_{n=0}^{\infty} C(n)_{\bullet+n}$$

and differential

$$D^{\prod} = \partial + \delta,$$

where δ is a signed sum of the coface maps; see, e.g., [Wan24, (2.4)]. Now, consider

$$C(n) = C(n;a,b) = R_a \otimes C^{\otimes n} \otimes R_b$$

with coface maps

$$\delta_i\colon R_a \otimes C^{\otimes n-1} \otimes R_b \to R_a \otimes C^{\otimes n} \otimes R_b \quad (0 \leq i \leq n)$$
$$c_0 \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes c_n \mapsto c_0 \otimes \cdots \otimes c_{i-1} \otimes \Delta(c_i) \otimes c_{i+1} \otimes \cdots \otimes c_n$$

induced by the coproduct $\Delta : C \to C \otimes C$ when $i = 1, \ldots, n-1$, and

$$\delta_0\colon c_0 \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes c_n \mapsto \rho_{r,a}(c_0) \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes c_n,$$
$$\delta_n\colon c_0 \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes c_n \mapsto c_0 \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes \rho_{l,b}(c_n),$$

where the map $\rho_{r,a}$ is defined as the composition

$$R_a \cong R \xrightarrow{\cong} R \otimes R \xrightarrow{\mathrm{id}\otimes i_a} R \otimes C \cong R_a \otimes C$$

for $i_a\colon R \to C$ being induced by the inclusion $\{a\} \hookrightarrow S$ and $\rho_{l,b}\colon R_b \to C \otimes R_b$ is defined similarly. The codegeneracy maps are defined by

$$\sigma_i\colon R_a \otimes C^{\otimes n+1} \otimes R_b \to R_a \otimes C^{\otimes n} \otimes R_b \quad (0 \leq i \leq n)$$
$$c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1} \otimes c_{n+2} \mapsto c_0 \otimes \cdots \otimes c_i \otimes \varepsilon(c_{i+1}) \otimes c_{i+2} \otimes \cdots \otimes c_{n+2}$$

where $\varepsilon : C \twoheadrightarrow C_0 \to R$ is the counit map. Then $\mathsf{Cobar}^{\prod}(C)(a,b)$ is defined as the totalization $\mathrm{Tot}^{\prod}(\{C(n;a,b)\}_{n\geq 0})$.

The *conormalization* of the totalization $\mathrm{Tot}^{\prod}(\{C(n;a,b)\}_{n\geq 0})$, which we denote by $N^c\mathrm{Tot}^{\prod}(\{C(n;a,b)\}_{n\geq 0})$, is the subcomplex where any σ_i vanishes. The inclusion $N^c\mathrm{Tot}^{\prod} \hookrightarrow \mathrm{Tot}^{\prod}$ is a quasi-isomorphism (cf. [Iri17, Lemma 2.5]).

Example 2.1. Let X be a topological space and let $b \in X$. We are interested in the following examples of differential graded coalgebras C equipped with a set of objects S such that $C_0 = R[S]$:

- $C = C_\bullet(X)$, with S the set of points in X.
- $C = C^0_\bullet(X, b)$ or $C^1_\bullet(X, b)$, with $S = \{b\}$.

In case $X = M$ is a smooth manifold, we may also restrict the discussion to piecewise smooth singular chains $C^{\mathrm{s}}_\bullet$.

2.2. **The natural transformations $\mathcal{A}, \mathcal{G}$ and the natural map It_X.** For any topological space X, the functors $\mathcal{A}_X$, $\mathcal{G}_X$ act as the identity on objects. In the following, we describe the action of $\mathcal{A}_X$, $\mathcal{G}_X$ and It_X on the morphism complexes.

2.2.1. *The "many object" version of Adams' map.* Denote by $v_0, \dots, v_n$ the vertices of the standard n-simplex $\mathbb{A}^n \subset \mathbb{R}^{n+1}$. Using the method of acyclic models, one may construct a collection of singular cubical chains

$$\{\theta_n \colon I^{n-1} \to \mathsf{P}(\mathbb{A}^n)(v_0, v_n)\}_{n \geq 1}$$

such that

(1) $\theta_1(0) \in \mathsf{P}(\mathbb{A}^1)(v_0, v_1)$ is the (Moore) path $\theta_1(0) \colon [0, \sqrt{2}] \to \mathbb{A}^1$ given by

$$\theta_1(0)(s) = v_0 + \frac{s}{\sqrt{2}}(v_1 - v_0),$$

and

(2)

$$\partial^\square \circ \theta_n = \sum_{i=1}^{n-1} (-1)^i (\mathsf{P}(l_{n-i,n}) \circ \theta_{n-i}) * (\mathsf{P}(f_{i,n}) \circ \theta_i) - \sum_{i=1}^{n-1} (-1)^i \mathsf{P}(d_i) \circ \theta_i,$$

where $l_{j,n} \colon \mathbb{A}^j \hookrightarrow \mathbb{A}^n$ and $f_{j,n} \colon \mathbb{A}^j \hookrightarrow \mathbb{A}^n$ denote the last and first j-dimensional face inclusions, respectively, $d_i \colon \mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^n$ denotes the i-th face inclusion, and $*$ denotes concatenation of paths.

See [MR24, Section 3.4] for a description of an explicit choice of maps $\{\theta_n\}_{n \geq 1}$. This construction goes back to [Ada56, Section 3].

Any such collection of maps $\{\theta_n\}_{n \geq 1}$ gives rise to a well defined linear map of degree -1

$$A_X \colon \overline{\mathcal{C}}(X) \to \bigoplus_{a,b \in X} \mathsf{C}^\square(\mathsf{P}(X))(a, b)$$

by sending a class represented by a singular chain $\sigma \colon \mathbb{A}^n \to X$ to

$$A_X(\sigma) = \mathsf{P}(\sigma) \circ \theta_n \colon I^{n-1} \to \mathsf{P}(X)(\sigma(v_0), \sigma(v_n)).$$

The map A_X satisfies the (curved) Maurer-Cartan equation

$$\partial^\square \circ A_X - A_X \circ \widetilde{\partial} = * \circ (A_X \otimes A_X) \circ \widetilde{\Delta} + \widetilde{h},$$

where $\widetilde{h} \colon \overline{\mathcal{C}}(X) \to \bigoplus_{a,b \in X} \mathsf{C}^\square(\mathsf{P}(X))(a, b)$ is the degree -2 map given by the composition

$$\mathcal{C}(X) \xrightarrow{\rho_r} \mathcal{C}(X) \otimes \mathcal{C}_0(X) \xrightarrow{h \otimes \mathrm{id}} R \otimes \mathcal{C}_0(X) \cong \mathcal{C}_0(X)$$
$$\hookrightarrow \bigoplus_{a \in X} \mathsf{C}^\square(\mathsf{P}(X))(a, a) \hookrightarrow \bigoplus_{a,b \in X} \mathsf{C}^\square(\mathsf{P}(X))(a, b).$$

Extending A_X to be compatible with composition, for any $a, b \in X$, we obtain a natural degree 0 linear map

$$\mathcal{A}_X\colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a,b) \to \mathsf{C}^{\square}(\mathsf{P}(X))(a,b),$$

which, by the Maurer-Cartan equation above, is a chain map.

2.2.2. *A formal dual of Chen's iterated integral map.* Given $a, b \in X$, the chain map

$$It_X\colon \mathsf{C}^{\square}(\mathsf{P}X)(a,b) \to \mathsf{Cobar}^{\prod}(C(X))(a,b) \tag{2.3}$$

is defined as the composition of several maps specified below.

Step 1. There is a natural chain map

$$\eta^{\square,\mathbb{A}}\colon \mathsf{C}^{\square}_{\bullet}(\mathsf{P}X)(a,b) \to \mathsf{C}^{\mathbb{A}}_{\bullet}(\mathsf{P}X)(a,b) = C_{\bullet}(\mathsf{P}X(a,b)),$$

induced by the standard triangulation of cubes. Concretely, for any singular cube $\lambda\colon [0,1]^n \to \mathsf{P}X(a,b)$,

$$\eta^{\square,\mathbb{A}}(\lambda) = \sum_{\tau\in S_n} (-1)^{\mathrm{sgn}(\tau)}\, \lambda\circ\iota_\tau,$$

where S_n denotes the symmetric group on n letters, and

$$\iota_\tau\colon \mathbb{A}^n \to [0,1]^n, \qquad (t_1,\dots,t_n) \mapsto (t_{\tau(1)},\dots,t_{\tau(n)})$$

for $0 \le t_1 \le \cdots \le t_n \le 1$. More generally,

$$\eta^{\square,\mathbb{A}}\colon C^{\square}_{\bullet} \Rightarrow C^{\mathbb{A}}_{\bullet}$$

is a natural transformation from the normalized cubical singular chain functor on topological spaces to the normalized simplicial singular chain functor; the map displayed above is its component at the space $\mathsf{P}X(a,b)$ and these are natural with respect to continuous maps of spaces.

Step 2. Consider the cosimplicial space

$$[n] \longmapsto \mathsf{P}X(a,b) \times \mathbb{A}^n$$

with cosimplicial structure induced from the standard one $[n] \mapsto \mathbb{A}^n$. Set

$$u_n := \mathrm{id}_{\mathbb{A}^n} \in C_n(\mathbb{A}^n), \quad u := \{u_n\}_{n\ge 0}.$$

Then, the linear maps

$$C_{\bullet}(\mathsf{P}X(a,b)) \to C_{\bullet+n}(\mathsf{P}X(a,b)\times\mathbb{A}^n), \quad x \mapsto (-1)^n x \times u_n,$$

for all $n \ge 0$ together induce a chain map

$$\Phi_u\colon C_{\bullet}(\mathsf{P}X(a,b)) \to \mathrm{Tot}^{\prod}_{\bullet}(\{C(\mathsf{P}X(a,b)\times\mathbb{A}^n)\}_{n\ge 0}).$$

More generally, any choice of $u = \{u_n \in C_n(\mathbb{A}^n)\}_{n\ge 0}$ that satisfies

$$[u_0] = [\mathrm{id}_{\mathbb{A}^0}] \in H_0(\mathbb{A}^0) \quad \text{and} \quad \partial u_n = \sum_{i=0}^{n} (-1)^i (d_i)_*(u_{n-1}) \quad (\forall n \ge 1) \tag{2.4}$$

suffices to define Φ_u. Moreover, by an acyclic models argument, using the fact that $H_{n+1}(\mathbb{A}^n) = 0$ for all $n \ge 0$, one can show that the chain homotopy class of Φ_u is independent of the choice of u.

Step 3. Consider the cosimplicial space $n \mapsto \{a\}\times X^n \times \{b\}$ with cofaces induced by the diagonal map and codegeneracies given by forgetful maps. The evaluation maps

$$\mathrm{Ev}_n\colon \quad \mathsf{P}X(a,b)\times\mathbb{A}^n \to \{a\}\times X^n\times\{b\},$$

$$((\gamma, T), (t_1, \dots, t_n)) \mapsto (a, \gamma(t_1 T), \dots, \gamma(t_n T), b)$$

for all $n \geq 0$ respect cosimplicial structures, inducing a chain map

$$\mathrm{Ev}_* \colon \mathrm{Tot}^{\prod}(\{C(\mathsf{P}X(a,b) \times \mathbb{A}^n)\}_{n\geq 0}) \to \mathrm{Tot}^{\prod}(\{C(\{a\} \times X^n \times \{b\})\}_{n\geq 0}).$$

Step 4. Iterations of the standard Alexander-Whitney map define a cosimplicial chain map

$$AW_n \colon C_\bullet(\{a\} \times X^n \times \{b\}) \to (R_a \otimes C(X)^{\otimes n} \otimes R_b)_\bullet, \quad n \geq 0,$$

inducing a chain map

$$AW \colon \mathrm{Tot}^{\prod}_\bullet(\{C(\{a\} \times X^n \times \{b\})\}_{n\geq 0}) \to \mathsf{Cobar}^{\prod}(C(X))(a,b).$$

We have thus defined (2.3) as the composition

$$It_X = AW \circ \mathrm{Ev}_* \circ \Phi_u \circ \eta^{\square,\mathbb{A}}.$$

Remark 2.2. In the definition of It_X, we chose to triangulate cubes right from the beginning. Alternatively, one may remain in the setting of normalized cubical singular chains and pass to normalized simplicial singular chains later, either before Ev_* or before AW. In this case,

$$It_X = AW \circ \eta^{\square,\mathbb{A}} \circ \mathrm{Ev}_* \circ \Phi_v = AW \circ \mathrm{Ev}_* \circ \eta^{\square,\mathbb{A}} \circ \Phi_v,$$

where $v = \{\, v_n \in C_n^{\square}(\mathbb{A}^n) \,\}_{n\geq 0}$ is chosen such that v_0 is the unique point map and

$$\partial^{\square} v_n = \sum_{i=0}^{n} (-1)^i (d_i)_*(v_{n-1}) \quad \text{for all } n \geq 1.$$

Lemma 2.3 below, together with the fact that $\eta^{\square,\mathbb{A}}$ intertwines the cross products on cubes and simplices, implies that these two approaches are equivalent:

$$\mathrm{Ev}_* \circ \Phi_u \circ \eta^{\square,\mathbb{A}} = \eta^{\square,\mathbb{A}} \circ \mathrm{Ev}_* \circ \Phi_v = \mathrm{Ev}_* \circ \eta^{\square,\mathbb{A}} \circ \Phi_v$$

if $v = \eta^{\mathbb{A},\square}(u)$ or $u = \eta^{\square,\mathbb{A}}(v)$. Note that $v = \eta^{\mathbb{A},\square}(u)$ acturally implies $u = \eta^{\square,\mathbb{A}}(v)$ by Lemma 2.3.

Lemma 2.3. There is a natural transformation

$$\eta^{\mathbb{A},\square} \colon C_\bullet^{\mathbb{A}} \Rightarrow C_\bullet^{\square}$$

from the normalized simplicial singular chain functor to the normalized cubical singular chain functor, such that

$$\eta^{\square,\mathbb{A}} \circ \eta^{\mathbb{A},\square} = \mathrm{id}_{C_\bullet^{\mathbb{A}}}.$$

Proof. For every $n \geq 0$, define a folding map

$$f_n \colon I^n \to \Delta^n, \qquad f_n(t_1, \dots, t_n) = (t_1', \dots, t_n'), \quad t_k' = \max\{t_1, \dots, t_k\}.$$

A similar map appears in [Igu09, Section 4.2], although for a different purpose.

For any topological space Y and any singular simplex $\sigma \colon \Delta^n \to Y$, define

$$\eta_Y^{\mathbb{A},\square}(\sigma) = \sigma \circ f_n \colon I^n \to Y,$$

and extend linearly to $C_\bullet^{\mathbb{A}}(Y)$. By construction, $\eta_Y^{\mathbb{A},\square}$ is natural in Y.

To see that $\eta_Y^{\mathbb{A},\square}$ is a chain map, observe that among the cubical faces of $\eta_Y^{\mathbb{A},\square}(\sigma)$, those of the form $\{t_k = 1\}$ with $k < n$ are degenerate, while the remaining $n+1$ non-degenerate faces correspond exactly to the simplicial faces of σ.

Finally, to verify that $\eta_Y^{\square,\mathbb{A}} \circ \eta_Y^{\mathbb{A},\square} = \mathrm{id}_{C_\bullet^{\mathbb{A}}(Y)}$, compute

$$(\eta_Y^{\square,\mathbb{A}} \circ \eta_Y^{\mathbb{A},\square})(\sigma) = \sum_{\tau \in S_n} \mathrm{sgn}(\tau)\, \sigma \circ f_n \circ \iota_\tau = \sigma,$$

since $f_n \circ \iota_\tau$ is degenerate whenever $\tau \neq \mathrm{id}$, and $f_n \circ \iota_{\mathrm{id}} = \mathrm{id}_{\Delta^n}$. □

Remark 2.4. Let M be a smooth manifold, $\Omega^\bullet(M)$ the dg algebra of differential forms on M, and $\mathsf{P}^{\mathrm{s}}M(a,b)$ the space of piecewise smooth Moore loops in M from a to b. Set $R = \mathbb{R}$. The iterated integral map for $\mathsf{P}^{\mathrm{s}}M(a,b)$, originally due to Chen in [Che73], is a cochain map

$$\int \colon \mathsf{Bar}(\mathbb{R}_a, \Omega(M), \mathbb{R}_b) \to \Omega^\bullet(\mathsf{P}^{\mathrm{s}}M(a,b)), \tag{2.5}$$

where Bar denotes a version of the classical bar construction and differential forms on $\mathsf{P}^{\mathrm{s}}M(a,b)$ are defined via the framework of of *differentiable spaces*. Following the presentation in [GJP91], $\int$ is induced by a sequence of maps

$$\mathbb{R}_a \otimes \Omega^\bullet(M)^{\otimes n} \otimes \mathbb{R}_b \hookrightarrow \Omega^\bullet(\{a\} \times M^n \times \{b\}) \xrightarrow{\mathrm{Ev}_n^*} \Omega^\bullet(\mathsf{P}^{\mathrm{s}}M(a,b) \times \mathbb{A}^n) \xrightarrow{\int_{\mathbb{A}^n}} \Omega^{\bullet-n}(\mathsf{P}^{\mathrm{s}}M(a,b))$$

for $n \geq 0$, where $\int_{\mathbb{A}^n}$ denotes integration along the fibers. Consider the pairing

$$\Omega^\bullet(Y) \times C_\bullet^{\mathrm{s}}(Y) \to \mathbb{R}$$

induced by integration, where Y is M or $\mathsf{P}^{\mathrm{s}}M(a,b)$, and the pairing

$$\begin{aligned} &\mathsf{Bar}(\mathbb{R}_a, \Omega^\bullet(M), \mathbb{R}_b) \times \mathsf{Cobar}^{\prod}(C_\bullet^{\mathrm{s}}(M))(a,b) \to \mathbb{R} \\ &\Big\langle \sum_{n=0}^{N} 1 \otimes \omega_{n,1} \otimes \cdots \otimes \omega_{n,n} \otimes 1\,,\ \big(1 \otimes \alpha_{m,1} \otimes \cdots \otimes \alpha_{m,m} \otimes 1\big)_{m \geq 0} \Big\rangle \\ &= \sum_{n=1}^{N} \langle \omega_{n,1}, \alpha_{n,1} \rangle \cdots \langle \omega_{n,n}, \alpha_{n,n} \rangle \end{aligned} \tag{2.6}$$

induced by summing up integrations levelwise. There is a chain map

$$It_M \colon C_\bullet^{\mathrm{s}}(\mathsf{P}^{\mathrm{s}}M(a,b)) \to \mathsf{Cobar}_\bullet^{\prod}(C^{\mathrm{s}}(M))(a,b)$$

defined analogously to (2.3). The maps It_M and $\int$ (2.5) are formally dual in the sense that

$$\Big\langle \int \boldsymbol{\omega}\,,\ \alpha \Big\rangle = \Big\langle \boldsymbol{\omega}\,,\ It_M(\alpha) \Big\rangle$$

for any $\boldsymbol{\omega} \in \mathsf{Bar}(\mathbb{R}_a, \Omega^\bullet(M), \mathbb{R}_b)$ and $\alpha \in C_\bullet^{\mathrm{s}}(\mathsf{P}^{\mathrm{s}}M(a,b))$.

Now set $a = b$, and denote the product on $C_\bullet^{\mathrm{s}}(\mathsf{P}^{\mathrm{s}}M(a,a))$ by $\times$. For brevity, write $1 \otimes \omega_1 \otimes \cdots \otimes \omega_n \otimes 1$ as $\omega_1 \cdots \omega_n$. For any $\alpha, \beta \in C_\bullet^{\mathrm{s}}(\mathsf{P}^{\mathrm{s}}M(a,a))$ and $\omega_1, \ldots, \omega_n \in \Omega(M)$, we have

$$\begin{aligned} \Big\langle \omega_1 \cdots \omega_n\,,\ It_M(\alpha \times \beta) \Big\rangle &= \Big\langle \int \omega_1 \cdots \omega_n\,,\ \alpha \times \beta \Big\rangle \\ &= \sum_{i=1}^{n} \Big\langle \int \omega_1 \cdots \omega_i\,,\ \alpha \Big\rangle \Big\langle \int \omega_{i+1} \cdots \omega_n\,,\ \beta \Big\rangle \\ &= \sum_{i=1}^{n} \Big\langle \omega_1 \cdots \omega_i\,,\ It_M(\alpha) \Big\rangle \Big\langle \omega_{i+1} \cdots \omega_n\,,\ It_M(\beta) \Big\rangle \end{aligned} \tag{2.7}$$

$$= \Big\langle \omega_1 \cdots \omega_n \, , \, It_M(\alpha) It_M(\beta) \Big\rangle,$$

where the equality on the second line is exactly [Che73, (1.6.2)], and the last line follows from the fact that the concatenation product on $\mathsf{Cobar}^{\prod}(C^{\mathrm{s}}_\bullet(M))(a,b)$ is dual to the deconcatenation coproduct on $\mathsf{Bar}(\mathbb{R}_a, \Omega^\bullet(M), \mathbb{R}_a)$ under the pairing (2.6).

Equation (2.7) suggests that It_M preserves the products (compositions) up to terms that vanish under the integration pairing with differential forms. A more general and precise statement is Theorem 1.2, which we prove in Section 3.1.

2.2.3. *The natural transformation* $\mathcal{G}$. For any $a, b \in X$, define a degree 0 linear map

$$G_X \colon R_a \boxtimes s^{-1}\overline{\mathcal{C}}(X) \boxtimes R_b \to \mathsf{Cobar}^{\prod}(C(X))(a,b)$$

such that for any $\sigma \colon \Delta^n \to X$,

$$G_X(s^{-1}\sigma) = s^{-1}\sigma + e(\sigma) \in (R_a \otimes s^{-1}C(X) \otimes R_b) \oplus (R_a \otimes R \otimes R_b),$$

where e is the map (2.2). Extending G_X to be compatible with compositions, and using the identification

$$R_a \boxtimes (s^{-1}\overline{\mathcal{C}}(X))^{\boxtimes 0} \boxtimes R_b \cong R \cong R_a \otimes R \otimes R_b = R_a \otimes (s^{-1}C(X))^{\otimes 0} \otimes R_b,$$

we obtain a natural degree 0 linear map

$$\mathcal{G}_X \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a,b) \to \mathsf{Cobar}^{\prod}(C(X))(a,b) \tag{2.8}$$

for any $a, b \in X$, which is compatible with compositions. A straightforward calculation shows that $\mathcal{G}_X$ is a chain map.

3. Construction of $\mathcal{I}$ (Theorem 1.2) and $\mathcal{H}$ (Theorem 1.4)

This section establishes the existence of a natural A_∞-transformation $\mathcal{I}$ extending $\{It_X\}_{X \in \mathsf{Top}}$, and of an A_∞-homotopy $\mathcal{H}$ between $\mathcal{F} = \mathcal{I} \circ \mathcal{A}$ and $\mathcal{G}$. Both $\mathcal{I}$ and $\mathcal{H}$ are constructed via the method of acyclic models; the construction of $\mathcal{I}$ is more geometric in nature, whereas the construction of $\mathcal{H}$ is more algebraic. At the end of this section we provide a precise formulation of the statement that $\mathcal{I}$ is a left A_∞-homotopy inverse of $\mathcal{A}$ for simply connected spaces, as announced in Remark 1.5.

We begin by fixing A_∞ sign conventions. Our sign conventions for A_∞-categories and A_∞-functors follow those for A_∞-algebras and A_∞-morphisms in [LV12] and differ from those in [Lef03]. The difference essentially arises from reversing the order of inserting objects in the multilinear maps, which, together with the Koszul sign convention for reordering graded objects, leads to a straightforward conversion rule between the two conventions. Applying this rule to the A_∞-algebra and A_∞-morphism signs in [Lef03] recovers exactly the conventions of [LV12]. Since [LV12] does not specify signs for A_∞-homotopies, we extend this conversion rule to the A_∞-homotopy signs from [Lef03], ensuring all signs remain compatible with [LV12].

For an A_∞-category C and objects $X_0, \ldots, X_n \in \mathrm{Ob}(\mathsf{C})$, the structure maps

$$m_n^{\mathsf{C}} \colon \mathrm{Hom}^{\mathsf{C}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}^{\mathsf{C}}(X_{n-1}, X_n) \to \mathrm{Hom}^{\mathsf{C}}(X_0, X_n)$$

are of degree $n-2$, satisfying the A_∞-relations

$$\sum_{p+q+r=n} (-1)^{p+qr} m^{\mathsf{C}}_{p+1+r} \circ \left(1^{\otimes p} \otimes m^{\mathsf{C}}_q \otimes 1^{\otimes r}\right) = 0, \quad \forall n \geq 1.$$

For two A_∞-categories C, C', an A_∞-functor $\mathsf{F} \colon \mathsf{C} \to \mathsf{C}'$ consists of a map

$$F \colon \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{C}')$$

on objects and a collection of degree $n-1$ linear maps

$$F_n\colon \mathrm{Hom}^{\mathsf{C}}(X_0,X_1)\otimes\cdots\otimes \mathrm{Hom}^{\mathsf{C}}(X_{n-1},X_n)\to \mathrm{Hom}^{\mathsf{C}'}\big(F(X_0),F(X_n)\big)$$

for all $n\geq 1$ and $X_0,\dots,X_n\in \mathrm{Ob}(\mathsf{C})$, satisfying that for any $n\geq 1$,

$$\sum_{p+q+r=n}(-1)^{p+qr}F_{p+1+r}\big(1^{\otimes p}\otimes m_q^{\mathsf{C}}\otimes 1^{\otimes r}\big)=\sum_{\substack{k\geq 1\\ i_1+\cdots+i_k=n}}(-1)^{\epsilon}m_k^{\mathsf{C}'}\big(F_{i_1}\otimes\cdots\otimes F_{i_k}\big),$$

where

$$\epsilon=\sum_{j=1}^{k}(k-j)(i_j-1).$$

For two A_∞-functors $\mathsf{F},\mathsf{F}'\colon\mathsf{C}\to\mathsf{C}'$ associated with the same object map F, an A_∞-homotopy $\mathsf{H}\colon\mathsf{F}\Rightarrow\mathsf{F}'$ consists of a collection of degree n linear maps

$$H_n\colon \mathrm{Hom}^{\mathsf{C}}(X_0,X_1)\otimes\cdots\otimes \mathrm{Hom}^{\mathsf{C}}(X_{n-1},X_n)\to \mathrm{Hom}^{\mathsf{C}'}\big(F(X_0),F(X_n)\big)$$

for all $n\geq 1$ and $X_0,\dots,X_n\in \mathrm{Ob}(\mathsf{C})$, satisfying that for any $n\geq 1$,

$$\begin{aligned}F_n-F_n'=&\sum_{p+q+r=n}(-1)^{p+qr}H_{p+1+r}\big(1^{\otimes p}\otimes m_q^{\mathsf{C}}\otimes 1^{\otimes r}\big)\\&+\sum_{\substack{k\geq l\geq 1\\ i_1+\cdots+i_k=n}}(-1)^{\delta}m_k^{\mathsf{C}'}\Big(G_{i_1}\otimes\cdots\otimes G_{i_{l-1}}\otimes H_{i_l}\otimes F_{i_{l+1}}\otimes\cdots\otimes F_{i_k}\Big),\end{aligned}$$

where

$$\delta=k-1+\sum_{j=1}^{k}(k-j)(i_j-1).$$

Under the above conventions, we focus on dg categories, i.e., A_∞-categories where m_1 is the differential, m_2 the composition, and $m_n=0$ for all $n>2$.

3.1. **Proof of Theorem 1.2.** Recall from Section 2.2.2 that It_X is the composition of natural chain maps

$$\begin{aligned}\mathsf{C}^{\square}_{\bullet}(\mathsf{P}X)(a,b)&\xrightarrow{\Phi_v}\mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathsf{P}X(a,b)\times\mathbb{A}^n)\}_{n\geq 0})\\&\xrightarrow{\eta^{\square,\mathbb{A}}}\mathrm{Tot}^{\prod}_{\bullet}(\{C(\mathsf{P}X(a,b)\times\mathbb{A}^n)\}_{n\geq 0})\\&\xrightarrow{\mathrm{Ev}_*}\mathrm{Tot}^{\prod}_{\bullet}(\{C(\{a\}\times X^n\times\{b\})\}_{n\geq 0})\xrightarrow{AW}\mathsf{Cobar}^{\prod}_{\bullet}(C(X))(a,b).\end{aligned}\tag{3.1}$$

Clearly, AW defines a functor between dg categories, natural in $X\in\mathsf{Top}$, where the composition on

$$\big\{\mathrm{Tot}^{\prod}_{\bullet}(\{C(\{a\}\times X^n\times\{b\})\}_{n\geq 0})\big\}_{a,b\in X}$$

is induced by the Cartesian product of spaces:

$$\{a\}\times X^{n_1}\times\{b\}\times\{b\}\times X^{n_2}\times\{c\}\to\{a\}\times X^{n_1+n_2}\times\{c\}.$$

Hence it suffices to show that $\mathrm{Ev}_*\circ\eta^{\square,\mathbb{A}}\circ\Phi_v$ is an A_∞-functor between dg categories. This naturally leads one to seek associative compositions on the families

$$\big\{\mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathsf{P}X(a,b)\times\mathbb{A}^n)\}_{n\geq 0})\big\}_{a,b\in X},\quad \big\{\mathrm{Tot}^{\prod}_{\bullet}(\{C(\mathsf{P}X(a,b)\times\mathbb{A}^n)\}_{n\geq 0})\big\}_{a,b\in X},$$

respectively, and to analyze Ev_*, $\eta^{\square,\mathbb{A}}$, and Φ_v separately. The most natural candidate for such compositions would be induced by space maps

$$\mathsf{P}X(a,b)\times\mathbb{A}^{n_1}\times\mathsf{P}X(b,c)\times\mathbb{A}^{n_2}\to\mathsf{P}X(a,c)\times\mathbb{A}^{n_1+n_2},$$

defined by concatenating paths and gluing points in $\mathbb{A}^{n_1}, \mathbb{A}^{n_2}$ proportionally to their respective path lengths. However, such maps fail to exist when both paths have zero length, so the composition is only partially defined.

To settle this issue, we introduce a cosimplicial space

$$[n] \mapsto \mathsf{P}^n X(a,b) := \big\{(\gamma, T, t_1, \dots, t_n) \mid (\gamma, T) \in \mathsf{P}X(a,b),\ 0 \le t_1 \le \cdots \le t_n \le T\big\}, \tag{3.2}$$

whose cosimplicial structure parallels that of both $\mathsf{P}X(a,b) \times \mathbb{A}^n$ and $\{a\} \times X^n \times \{b\}$; see also [Wan24, (4.2a), (4.2b)]. The families

$$\big\{\mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathsf{P}^n X(a,b))\}_{n\ge 0})\big\}_{a,b\in X}, \quad \big\{\mathrm{Tot}^{\prod}_{\bullet}(\{C(\mathsf{P}^n X(a,b))\}_{n\ge 0})\big\}_{a,b\in X}$$

both admit associative compositions induced by path concatenation together with the obvious gluing of marked points, and therefore form the morphism sets of dg categories. Recognizing $[n] \mapsto \mathsf{P}^n X$ as a cosimplicial analogue of the nerve $\mathsf{N}(\mathsf{P}X)$ of $\mathsf{P}X$, we denote the resulting dg categories by

$$(\mathsf{Tot}\, C^{\square} \circ \mathsf{cN})(\mathsf{P}X) \quad \text{and} \quad (\mathsf{Tot}\, C^{\mathbb{A}} \circ \mathsf{cN})(\mathsf{P}X).$$

There is a family of natural maps

$$\begin{aligned} q_n \colon \mathsf{P}X(a,b) \times \mathbb{A}^n &\to \mathsf{P}^n X(a,b) \\ (\gamma, T, t_1, \dots, t_n) &\mapsto (\gamma, T, t_1 T, \dots, t_n T) \end{aligned}$$

respecting cosimplicial structures, and a family of evaluation maps

$$\begin{aligned} \mathrm{ev}_n \colon \mathsf{P}^n X(a,b) &\to \{a\} \times X^n \times \{b\} \\ (\gamma, T, t_1, \dots, t_n) &\mapsto (a, \gamma(t_1), \dots, \gamma(t_n), b) \end{aligned}$$

respecting cosimplicial structures and compositions, such that $\mathrm{Ev}_n = \mathrm{ev}_n \circ q_n$. Thus, we have a commutative diagram of natural chain maps:

$$\begin{array}{ccc} \mathsf{C}^{\square}_{\bullet}(\mathsf{P}X)(a,b) & \xrightarrow{q_* \circ \Phi_v} & \mathrm{Tot}^{\prod}_{\bullet}\big(\{C^{\square}(\mathsf{P}^n X(a,b))\}_{n\ge 0}\big) \\ \Phi_v \downarrow & & \downarrow \eta^{\square,\mathbb{A}} \\ \mathrm{Tot}^{\prod}_{\bullet}\big(\{C^{\square}(\mathsf{P}X(a,b) \times \mathbb{A}^n)\}_{n\ge 0}\big) & & \mathrm{Tot}^{\prod}_{\bullet}\big(\{C(\mathsf{P}^n X(a,b))\}_{n\ge 0}\big) \\ \downarrow \eta^{\square,\mathbb{A}} & & \downarrow \mathrm{ev}_* \\ \mathrm{Tot}^{\prod}_{\bullet}\big(\{C(\mathsf{P}X(a,b) \times \mathbb{A}^n)\}_{n\ge 0}\big) & \xrightarrow{\mathrm{Ev}_*} & \mathrm{Tot}^{\prod}_{\bullet}(\{C(\{a\} \times X^n \times \{b\})\}_{n\ge 0}), \end{array} \tag{3.3}$$

where ev_* defines a functor between dg categories.

Now, Theorem 1.2 reduces to the following lemma.

Lemma 3.1. For any topological space X, the natural chain map

$$\mathcal{I}_{X,1} := \eta^{\square,\mathbb{A}} \circ q_* \circ \Phi_v$$

in (3.3) extends to a natural A_∞-functor

$$\mathcal{I}_X = \{\mathcal{I}_{X,k}\}_{k\ge 1} \colon \mathsf{C}^{\square}(\mathsf{P}X) \to (\mathsf{Tot}\, C^{\mathbb{A}} \circ \mathsf{cN})(\mathsf{P}X).$$

Explicitly, there exists, for each $k \ge 1$ and $a_0, \dots, a_k \in X$, a natural linear map

$$\mathcal{I}_{X,k} \colon \bigotimes_{i=0}^{k} \mathsf{C}^{\square}(\mathsf{P}X)(a_{i-1}, a_i) \to \mathrm{Tot}^{\prod}\big(\{C(\mathsf{P}^n X(a_0, a_k))\}_{n\ge 0}\big)$$

of degree $k-1$; and the collection $\{\mathcal{J}_{X,k}\}_{k\geq 1}$ satisfies, for any $k \geq 1$,

$$\begin{aligned} D^{\prod} \circ \mathcal{J}_{X,k} \ &= \sum_{p+r=k-1} (-1)^{k-1} \mathcal{J}_{X,k} \circ (1^{\otimes p} \otimes \partial^{\square} \otimes 1^{\otimes r}) \\ &\quad + \sum_{p+r=k-2} (-1)^{p} \mathcal{J}_{X,k-1} \circ (1^{\otimes p} \otimes \mu \otimes 1^{\otimes r}) \\ &\quad + \sum_{i+j=k} (-1)^{i} \mu^{\prod} \circ (\mathcal{J}_{X,i} \otimes \mathcal{J}_{X,j}), \end{aligned}$$

where μ and $\mu^{\prod}$ denote the respective compositions.

Once Lemma 3.1 is proved, Theorem 1.2 follows by setting

$$\mathfrak{I}_X := AW \circ \mathrm{ev}_* \circ \mathcal{J}_X.$$

In preparation for the proof of Lemma 3.1, for any $T \geq 0$ and integer $n \geq 0$, set

$$\mathbb{\Delta}^n_T := \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq T\},$$

which is the standard n-simplex scaled by T. In particular, $\mathbb{\Delta}^n_0$ consists of a single point 0. For any $T \geq 0$, the assignment $[n] \mapsto \mathbb{\Delta}^n_T$ forms a cosimplicial space analogous to the standard one at $T = 1$. Indeed, $\{\mathbb{\Delta}^n_T\}_{n\geq 0}$ identifies with the cosimplicial subspace of $\{\mathsf{P}^n\{a\}(a,a)\}_{n\geq 0}$ consisting of constant marked paths of length T in the singleton space $\{a\}$. The family $\{\mathbb{\Delta}^n_T\}_{n\geq 0,\, T\geq 0}$ inherits an associative composition

$$\circ : \mathbb{\Delta}^{n_1}_{T_1} \times \mathbb{\Delta}^{n_2}_{T_2} \to \mathbb{\Delta}^{n_1+n_2}_{T_1+T_2}$$

from $\{\mathsf{P}^n\{a\}(a,a)\}_{n\geq 0}$, inducing an associative composition

$$\circ : \mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathbb{\Delta}^n_{T_1})\}_{n\geq 0}) \otimes \mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathbb{\Delta}^n_{T_2})\}_{n\geq 0}) \to \mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathbb{\Delta}^n_{T_1+T_2})\}_{n\geq 0}),$$

which is a chain map.

The sequence $v = \{v_n \in C^{\square}_n(\mathbb{\Delta}^n)\}_{n\geq 0}$ in Remark 2.2 identifies with a 0-cycle

$$\tilde{v} = (\tilde{v}_n)_{n\geq 0} = ((-1)^n v_n)_{n\geq 0} \in \mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathbb{\Delta}^n)\}_{n\geq 0})$$

such that v_0 is the fudamental cycle of $\mathbb{\Delta}^0$. For any $T \geq 0$, denote by

$$s_T \colon \mathbb{\Delta}^n \to \mathbb{\Delta}^n_T$$

the scaling map, and define

$$\xi^0\langle T\rangle := (s_T)_*(\tilde{v}) \in \mathrm{Tot}^{\prod}_0(\{C^{\square}(\mathbb{\Delta}^n_T)\}_{n\geq 0}).$$

Lemma 3.2. There exists a family of chains

$$\Big\{\xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle \in \mathrm{Tot}^{\prod}_k\big(\{C^{\square}(\mathbb{\Delta}^n_{\lambda_0+\cdots+\lambda_k})\}_{n\geq 0}\big)\Big\}_{k\geq 0,\, \lambda_0,\ldots,\lambda_k\geq 0}$$

with $\xi^0\langle\lambda_0\rangle$ defined above, satisfying the following two properties:

(1) For any $k \geq 0$ and $\lambda_0, \ldots, \lambda_k \in [0, \infty)$, the chain $\xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle$ depends continuously on $\lambda_0, \ldots, \lambda_k$. More precisely, for any $k, n \geq 0$, there exists an integer $N(k,n) \geq 1$, and for each $1 \leq i \leq N(k,n)$ a scalar $c^k_{n,i} \in R$ and a continuous map

$$\tau^k_{n,i} \colon I^{k+n} \times [0,\infty)^{k+1} \to \mathbb{R}^n,$$

such that: for any $x \in I^{k+n}$ and $\lambda_0, \dots, \lambda_k \in [0,\infty)$, $\tau^k_{n,i}(x, \lambda_0, \dots, \lambda_k)$ lies in $\mathbb{\Delta}^n_{\lambda_0+\cdots+\lambda_k}$; and for any $\lambda_0, \dots, \lambda_k \in [0,\infty)$, the n-component of $\xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle$ is given by

$$\xi^k_n\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle = \sum_{i=1}^{N(k,n)} c^k_{n,i}\tau^k_{n,i}(\,\cdot\,, \lambda_0, \dots, \lambda_k).$$

(2) For any $k \geq 0$ and $\lambda_0, \dots, \lambda_k \geq 0$,

$$\begin{aligned} D^{\prod}(\xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle) &= \sum_{i=0}^{k-1}(-1)^i\xi^{k-1}\langle\lambda_0 \mid \cdots \mid \lambda_i + \lambda_{i+1} \mid \cdots \mid \lambda_k\rangle \\ &\quad + \sum_{i=0}^{k-1}(-1)^{i-1}\xi^i\langle\lambda_0 \mid \cdots \mid \lambda_i\rangle \circ \xi^{k-1-i}\langle\lambda_{i+1} \mid \cdots \mid \lambda_k\rangle. \end{aligned} \tag{3.4}$$

Proof. We prove the lemma by induction on k. Once a choice of ξ^k has been made for some k, it will remain fixed in all subsequent steps.

First, for $k = 0$, $\xi^0\langle\lambda_0\rangle = (s_{\lambda_0})_*(\tilde{v})$ depends continuously on λ_0, and satisfies

$$D^{\prod}(\xi^0\langle\lambda_0\rangle) = (s_{\lambda_0})_*(D^{\prod}\tilde{v}) = 0.$$

Next, assume the lemma holds for all $k' < k$ with $k \geq 1$, and let

$$\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle$$

denote the right-hand side of (3.4). By the inductive hypothesis, a straightforward computation shows that

$$D^{\prod}(\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle) = 0,$$

so $\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle$ is a cycle.

We now construct a bounding chain of $\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle$ that depends continuously on $\lambda_0, \dots, \lambda_k$. For $T = \lambda_0 + \cdots + \lambda_k$, consider the maps

$$h_n\colon [0,1] \times \mathbb{\Delta}^n_T \to \mathbb{\Delta}^n_T, \quad (s, t_1, \dots, t_n) \mapsto (st_1, \dots, st_n), \quad n \geq 0.$$

These define deformation retractions of $\mathbb{\Delta}^n_T$ onto the point $\mathbb{\Delta}^n_0 \subset \mathbb{\Delta}^n_T$, compatible with the cosimplicial structure maps.

Define a linear homotopy operator

$$\begin{aligned} H = \{H_n\}_{n\geq 0}\colon \mathrm{Tot}^{\prod}_{\bullet}(\{C^{\square}(\mathbb{\Delta}^n_T)\}_{n\geq 0}) &\to \mathrm{Tot}^{\prod}_{\bullet+1}(\{C^{\square}(\mathbb{\Delta}^n_T)\}_{n\geq 0}) \\ (x_n)_{n\geq 0} &\mapsto \big((h_n)_*(\mathrm{id}_{[0,1]} \times x_n)\big)_{n\geq 0}. \end{aligned}$$

Then

$$D^{\prod} \circ H + H \circ D^{\prod} = \mathrm{id} - \mathrm{c},$$

where $\mathrm{c} = \{\mathrm{c}_n\}_{n\geq 0}$ is induced by the contraction maps $h_n(\cdot, 0)$. Thus,

$$D^{\prod}\big(H(\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle)\big) = \Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle - \mathrm{c}(\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle).$$

It remains to show that

$$\mathrm{c}(\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle) = 0;$$

then $H(\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle)$ gives the desired bounding chain.

Since $h_n(\cdot, 0)$ factors through $\mathbb{\Delta}^n_0$, we can view each component

$$\mathrm{c}_n(\Xi^k_n\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle), \quad n \geq 0,$$

of the $(k-1)$-chain $\mathrm{c}(\Xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle)$ as lying in $C^{\square}_\bullet(\mathbb{A}^n_0) = C^{\square}_\bullet(\mathrm{pt})$, which vanishes in positive degrees. Since each such component has degree $k-1+n$, it vanishes automatically if $k > 1$ or $n > 0$; for the exceptional case $k = 1$ and $n = 0$, we have

$$\begin{aligned}\mathrm{c}_0(\Xi^1_0\langle\lambda_0 \mid \lambda_1\rangle) &= \mathrm{c}_0(\xi^0_0\langle\lambda_0+\lambda_1\rangle - \xi^0_0\langle\lambda_0\rangle \circ \xi^0_0\langle\lambda_1\rangle)\\ &= \mathrm{c}_0((s_{\lambda_0+\lambda_1})_*(v_0)) - \mathrm{c}_0((s_{\lambda_0})_*(v_0) \circ (s_{\lambda_1})_*(v_0)),\end{aligned}$$

which is the difference of two terms both equal to the fundamental cycle of $\mathbb{A}^0_0 = \mathrm{pt}$, and is therefore zero. This completes the proof. □

Proof of Lemma 3.1. Fix once and for all the family $\{\xi^k\langle\lambda_0 \mid \cdots \mid \lambda_n\rangle\}$, or equivalently, a collection of scalars and continuous maps

$$\{c^k_{n,i} \in R,\, \tau^k_{n,i} \colon I^{k+n} \times [0,\infty)^{k+1} \to \mathbb{R}^n\}_{k\geq 0,\, n\geq 0,\, 1\leq i\leq N(k,n)},$$

provided by Lemma 3.2.

Let $k \geq 1$, and consider points $a_0, \dots, a_k \in X$ together with singular cubes

$$\sigma_j \colon I^{d_j} \to \mathsf{P}X(a_{j-1}, a_j), \quad x \mapsto (\gamma_j(x), T_j(x)), \quad 1 \leq j \leq k,$$

where $T_j \colon I^{d_j} \to [0,\infty)$ and $\gamma_j(x) \colon [0, T_j(x)] \to X$ is a path.

For any $n \geq 0$ and $1 \leq i \leq N(k-1,n)$, define a continuous map

$$\begin{aligned}J_{k,n,i} \colon I^{d_1} \times \cdots \times I^{d_k} \times I^{k-1+n} &\to \mathsf{P}^n X(a_0, a_k)\\ (x_1, \dots, x_k, y) &\mapsto \big(\sigma_1(x_1) * \cdots * \sigma_k(x_k),\, \tau^{k-1}_{n,i}(y, T_1(x_1), \dots, T_k(x_k))\big).\end{aligned}$$

Each $J_{k,n,i}$ determines a normalized cubical chain on $\mathsf{P}^n X(a_0, a_k)$.

We then define, for every $k \geq 1$, an R-linear map

$$\mathcal{J}'_{X,k} \colon \bigotimes_{1\leq j\leq k} \mathsf{C}^{\square}(\mathsf{P}X)(a_{j-1}, a_j) \to \mathrm{Tot}^{\Pi}\big(\{C^{\square}(\mathsf{P}^n X(a_0, a_k))\}_{n\geq 0}\big)$$

$$\mathcal{J}'_{X,k}(\sigma_1 \otimes \cdots \otimes \sigma_k) = \left(\sum_{i=1}^{N(k-1,n)} (-1)^{(k-1)(d_1+\cdots+d_k)} c^{k-1}_{n,i} J_{k,n,i} \right)_{n\geq 0},$$

and finally define

$$\mathcal{J}_{X,k} = \eta^{\square,\mathbb{A}} \circ \mathcal{J}'_{X,k}.$$

By construction, $\{\mathcal{J}_{X,k}\}_{k\geq 1}$ is an A_∞-functor extending $\mathcal{J}_{X,1} = \eta^{\square,\mathbb{A}} \circ q_* \circ \Phi_v$. □

Remark 3.3. There is a subtlety: the collection $\{\mathcal{J}'_{X,k}\}_{k\geq 1}$ is *not* an A_∞-functor extending $\mathcal{J}'_{X,1} = q_* \circ \Phi_v$, as the A_∞-functor relations hold only up to switching cube coordinates in normalized cubical singular chains. To illustrate, consider the relation between $\mathcal{J}'_{X,1}$ and $\mathcal{J}'_{X,2}$. For $\sigma_i \colon I^{d_i} \to \mathsf{P}X(a_{i-1}, a_i)$, one needs to compare

$$D^{\Pi}(\mathcal{J}'_{X,2}(\sigma_1 \otimes \sigma_2)) + \mathcal{J}'_{X,2}(\partial(\sigma_1 \otimes \sigma_2))$$

with

$$\mathcal{J}'_{X,1}(\sigma_1 \circ \sigma_2) - \mathcal{J}'_{X,1}(\sigma_1) \circ \mathcal{J}'_{X,1}(\sigma_2).$$

For any $n \geq 0$, the $C^{\square}(\mathsf{P}^n X(a_0, a_2))$-component of $\mathcal{J}'_{X,1}(\sigma_1) \circ \mathcal{J}'_{X,1}(\sigma_2)$ is a weighted sum, over $n_1 + n_2 = n$ and $1 \leq i \leq N(0,n)$, of chains of the form

$$\begin{aligned}&I^{d_1} \times I^{n_1} \times I^{d_2} \times I^{n_2} \to \mathsf{P}^n X(a_0, a_2),\\ &(x_1, y_1, x_2, y_2) \mapsto \Big(\sigma_1(x_1) * \sigma_2(x_2),\, \tau^0_{n,i}(y_1, T_1(x_1)),\, T_1(x_1) + \tau^0_{n,i}(y_2, T_2(x_2))\Big).\end{aligned}$$

The corresponding part in $D^{\Pi}(\mathcal{J}'_{X,2}(\sigma_1 \otimes \sigma_2)) + \mathcal{J}'_{X,2}(\partial(\sigma_1 \otimes \sigma_2))$ is, however, a weighted sum of chains of the form

$$I^{d_1} \times I^{d_2} \times I^{n_1} \times I^{n_2} \to \mathsf{P}^n X(a_0, a_2),$$
$$(x_1, x_2, y_1, y_2) \mapsto \Big(\sigma_1(x_1) * \sigma_2(x_2),\, \tau^0_{n,i}(y_1, T_1(x_1)),\, T_1(x_1) + \tau^0_{n,i}(y_2, T_2(x_2))\Big).$$

Thus, the two expressions differ by switching the I^{d_2} and I^{n_1} coordinates, with signs changing accordingly. This difference vanishes upon passing to normalized simplicial singular chains, since $\eta^{\square,\triangle}$ is defined by a sum over all permutations of the cube coordinates, each with its signature.

Remark 3.4. By construction, $\xi^k\langle\lambda_0 \mid \cdots \mid \lambda_k\rangle = 0$ if $k > 0$ and some $\lambda_i = 0$. For $k = 0$, we have $\xi^0_n\langle 0\rangle = 0$ $(n > 0)$ and $\xi^0_0 = \mathrm{id}_{\mathrm{pt}} \in C^{\square}_0(\triangle^0_0) = C^{\square}_0(\mathrm{pt})$. Consequently, $\mathcal{J}_X$ is a unital A_∞-functor, in the sense that $\mathcal{J}_{X,1}$ respects identity morphisms and $\mathcal{J}_{X,k}$ $(k > 1)$ vanishes whenever one of its inputs is an identity morphism. It follows that the A_∞-functors $\mathcal{I}_X$ and $\mathcal{F}_X$ are also unital.

3.2. **Proof of Theorem 1.4.** We need the following acyclicity lemma.

Lemma 3.5. For any contractible topological space X and $a, b \in X$, the projection chain map

$$\mathrm{pr}_0\colon \mathsf{Cobar}^{\Pi}(C(X))(a, b) \to R_a \otimes R_b, \qquad (x_n)_{n\geq 0} \mapsto x_0$$

is a quasi-isomorphism. Hence, $H_\bullet(\mathsf{Cobar}^{\Pi}(C(X))(a, b)) \cong R$ is concentrated in degree zero.

Proof. This is an easy consequence of [Iri17, Lemma 8.3]. □

Now we begin the proof of Theorem 1.4. Explicitly, we need to show that there exists, for any topological space X, $k \geq 1$ and $a_0, \ldots, a_k \in X$, a natural linear map

$$\mathcal{H}_{X,k}\colon \bigotimes_{1\leq i\leq k} \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a_{i-1}, a_i) \to \mathsf{Cobar}^{\Pi}(C(X))(a_0, a_k),$$

and the collection $\{\mathcal{H}_{X,k}\}_{k\geq 1}$ satisfies, for any $k \geq 1$,

$$\begin{aligned}
\mathcal{F}_{X,k} - \mathcal{G}_{X,k} &= D^{\Pi} \circ \mathcal{H}_{X,k} + \sum_{p+r=k-1} (-1)^{k-1}\, \mathcal{H}_{X,k} \circ \big(1^{\otimes p} \otimes D^{\boxtimes} \otimes 1^{\otimes r}\big) \\
&\quad + \sum_{p+r=k-2} (-1)^p\, \mathcal{H}_{X,k-1} \circ \big(1^{\otimes p} \otimes \mu^{\boxtimes} \otimes 1^{\otimes r}\big) \\
&\quad + \sum_{i+j=k} (-1)^i \Big(\mu^{\Pi} \circ \big(\mathcal{H}_{X,i} \otimes \mathcal{F}_{X,j}\big) + \mu^{\Pi} \circ \big(\mathcal{G}_{X,i} \otimes \mathcal{H}_{X,j}\big)\Big),
\end{aligned} \tag{3.5}$$

where $\mu^{\boxtimes}$ and μ^{Π} denote the respective compositions in $\mathsf{Cobar}^{\boxtimes}$ and Cobar^{Π}.

Consider the following statement $\mathrm{S}(k, d)$ for integers $k \geq 1$ and $d \geq 0$:

- $\mathrm{S}(k, d)$: For any topological space X and $a_0, \ldots, a_k \in X$, the natural linear maps $\mathcal{H}_{X,k'}$ can be defined on all tensors
$$x_1 \otimes \cdots \otimes x_{k'} \in \bigotimes_{i=1}^{k'} \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a_{i-1}, a_i)$$
with $k' < k$ and $\deg x_1 + \cdots + \deg x_{k'} \leq d$, in such a way that (3.5) holds on these elements.

Theorem 1.4 is equivalent to the assertion that $\mathrm{S}(k,d)$ holds for all $k \geq 1$ and $d \geq 0$.

First of all, we set $\mathcal{H}_{X,k}$ to be zero if any of its inputs is a multiple of an identity morphism; then (3.5) holds on such inputs by Remark 3.4 and the fact that $\mathcal{G}_X$ respects identities.

Next, we prove $\mathrm{S}(k,d)$ for all k,d by induction on (k,d) in lexicographic order:

$$(k',d') < (k,d) \quad \text{if either (i) } k' < k, \text{ or (ii) } k' = k \text{ and } d' < d.$$

For the induction, we denote by $\mathcal{H}^d_{X,k}$ the restriction of the (yet to be constructed) map $\mathcal{H}_{X,k}$ to elements of total internal degree $\leq d$; similarly, we write $\mathcal{F}^d_{X,k}$ and $\mathcal{G}^d_X$ for the corresponding restrictions of $\mathcal{F}_{X,k}$ and $\mathcal{G}_X$. Once defined for a pair (k,d), the maps $\mathcal{H}^d_{X,k}$ are not modified in any subsequent steps.

Fix (k,d) with $k \geq 0$ and $d \geq 0$, suppose $\mathrm{S}(k',d')$ holds for all $(k',d') < (k,d)$, we aim to construct $\mathcal{H}^d_{X,k}$ with the desired properties. (Note that the assumption for $(k,d) = (1,0)$ is vacuumly true.) It suffices to define

$$\mathcal{H}^d_{X,k}(x_1 \otimes \cdots \otimes x_k)$$

for all

$$x_i \in R_{a_{i-1}} \boxtimes (s^{-1}\overline{\mathcal{C}(X)})^{\boxtimes n_i} \boxtimes R_{a_i},\ n_i > 0\ (1 \leq i \leq k)$$

with

$$\deg x_1 + \cdots + \deg x_k = d$$

in a way that is natural in X, and to verify that on such $x_1 \otimes \cdots \otimes x_k$,

$$\begin{aligned} D^{\Pi} \circ \mathcal{H}^d_{X,k} &= \mathcal{F}^d_{X,k} - \mathcal{G}^d_{X,k} + \sum_{p+r=k-1} (-1)^k\, \mathcal{H}^{d-1}_{X,k} \circ \big(1^{\otimes p} \otimes D^{\boxtimes} \otimes 1^{\otimes r}\big) \\ &+ \sum_{p+r=k-2} (-1)^{p-1}\, \mathcal{H}^d_{X,k-1} \circ \big(1^{\otimes p} \otimes \mu^{\boxtimes} \otimes 1^{\otimes r}\big) \\ &+ \sum_{i+j=k} (-1)^{i-1} \Big(\mu^{\Pi} \circ \big(\mathcal{H}^d_{X,i} \otimes \mathcal{F}^d_{X,j}\big) + \mu^{\Pi} \circ \big(\mathcal{G}^d_{X,i} \otimes \mathcal{H}^d_{X,j}\big)\Big). \end{aligned} \tag{3.6}$$

For any ordered k-tuple $\vec{\boldsymbol{d}}$ of ordered tuples of positive integers, $\vec{\boldsymbol{d}} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_k)$ with $\boldsymbol{d}_i = (d_{i,1}, \ldots, d_{i,n_i})$ and $d_{i,j} > 0$, define its degree by

$$\deg \vec{\boldsymbol{d}} \coloneqq \sum_{i=1}^{k} \sum_{j=1}^{n_i} (d_{i,j} - 1).$$

Let

$$\iota_{i,j} \colon \mathbb{\Delta}^{d_{i,j}} \hookrightarrow \bigvee_{i'=1}^{k} \bigvee_{j'=1}^{n_{i'}} \mathbb{\Delta}^{d_{i',j'}} =: \mathbb{\Delta}^{\vec{\boldsymbol{d}}}, \qquad 1 \leq i \leq k,\ 1 \leq j \leq n_i$$

be the obvious inclusion maps, where the wedge sum denotes a k-*necklace*, i.e., it is formed by successively identifying the last vertex of each simplex with the first vertex of the next in the order

$$\mathbb{\Delta}^{d_{1,1}} \vee \cdots \vee \mathbb{\Delta}^{d_{1,n_1}} \vee \mathbb{\Delta}^{d_{2,1}} \vee \cdots \vee \mathbb{\Delta}^{d_{k,n_k}}.$$

There is a canonical chain

$$\otimes_{i=1}^{k} \boxtimes_{j=1}^{n_i} s^{-1}\iota_{i,j} \in \bigotimes_{i=1}^{k} \Big(R_{v_{m_{i-1}}} \boxtimes \Big(\bigboxtimes_{j=1}^{n_i} s^{-1}\mathcal{C}_{d_{i,j}}(\mathbb{\Delta}^{\vec{\boldsymbol{d}}})\Big) \boxtimes R_{v_{m_i}}\Big),$$

where $m_i = \sum_{i'=1}^{i} \sum_{j=1}^{n_{i'}} d_{i',j}$, and v_p denotes the $(p+1)$-th vertex of $\mathbb{\Delta}^{\vec{\boldsymbol{d}}}$.

Now assume $\deg \vec{\boldsymbol{d}} = d$. Denote by

$$\Psi^{\vec{\boldsymbol{d}}} \in \mathsf{Cobar}^{\Pi}(C(\mathbb{\Delta}^{\vec{\boldsymbol{d}}}))(v_0, v_d)$$

the chain obtained by applying the right-hand side of (3.6) to $\otimes_{i=1}^{k} \boxtimes_{j=1}^{n_i} s^{-1}\iota_{i,j}$. Then we have the following lemma.

Lemma 3.6. There exists a chain $\psi^{\vec{\boldsymbol{d}}} \in \mathsf{Cobar}^{\Pi}(C(\mathbb{\Delta}^{\vec{\boldsymbol{d}}}))(v_0, v_d)$ such that

$$D^{\Pi}\psi^{\vec{\boldsymbol{d}}} = \Psi^{\vec{\boldsymbol{d}}}.$$

Proof. One checks by the inductive hypothesis that $\Psi^{\vec{\boldsymbol{d}}}$ is a cycle, so it remains to show that $[\Psi^{\vec{\boldsymbol{d}}}] = 0$ in homology.

Notice that $\deg \Psi^{\vec{\boldsymbol{d}}} = k - 1 + \sum_{i,j}(d_{i,j} - 1)$. If $k > 1$ or some $d_{i,j} > 1$, then $\deg \Psi^{\vec{\boldsymbol{d}}} > 0$, and then by Lemma 3.5, $[\Psi^{\vec{\boldsymbol{d}}}] = 0$.

It remains to consider the case $k = 1$ and $\vec{\boldsymbol{d}} = \boldsymbol{d}_1 = (1, \dots, 1)$. For simplicity, denote by

$$\iota_i \colon \mathbb{\Delta}^1 \hookrightarrow \mathbb{\Delta}^1 \vee \cdots \vee \mathbb{\Delta}^1 = \mathbb{\Delta}^{1,\dots,1}$$

the inclusion map sending $\mathbb{\Delta}^1$ onto the i-th copy of $\mathbb{\Delta}^1$. By definition,

$$\Psi^{1,\dots,1} = \mathcal{F}_{\mathbb{\Delta}^{1,\dots,1},1}(s^{-1}\iota_1 \boxtimes \cdots \boxtimes s^{-1}\iota_k) - \mathcal{G}_{\mathbb{\Delta}^{1,\dots,1}}(s^{-1}\iota_1 \boxtimes \cdots \boxtimes s^{-1}\iota_k).$$

Since $\mathcal{F}_X$, $\mathcal{G}_X$ are functorial in homology,

$$[\Psi^{1,\dots,1}] = \Pi_{i=1}^{k}[\mathcal{F}_{\mathbb{\Delta}^{1,\dots,1},1}(s^{-1}\iota_i)] - \Pi_{i=1}^{k}[\mathcal{G}_{\mathbb{\Delta}^{1,\dots,1}}(s^{-1}\iota_i)].$$

Since $\mathcal{F} = \mathcal{I} \circ \mathcal{A}$, for each $1 \le i \le k$, we have

$$\mathcal{F}_{\mathbb{\Delta}^{1,\dots,1},1}(s^{-1}\iota_i) = \mathcal{I}_{\mathbb{\Delta}^{1,\dots,1},1}(\bar{\iota_i}) = \big((v_{i-1} \cdot 1_R) \otimes (s^{-1}\overleftarrow{\iota_i})^{\otimes n} \otimes (1_R \cdot v_i)\big)_{n \ge 0},$$

where $\bar{\iota_i} \in \mathsf{P}\mathbb{\Delta}^{1,\dots,1}(v_{i-1}, v_i)$ denotes the point representing the path ι_i, and $\overleftarrow{\iota_i}$ is the reversal of ι_i, i.e. $\overleftarrow{\iota_i}(t) = \iota_i(1-t)$ for $t \in [0,1]$. Also,

$$\mathcal{G}_{\mathbb{\Delta}^{1,\dots,1}}(s^{-1}\iota_i) = v_{i-1} \cdot 1_R \cdot v_i + (v_0 \cdot 1_R) \otimes (s^{-1}\iota_i) \otimes (1_R \cdot v_1).$$

It follows that

$$(\mathrm{pr}_0)_*([\Psi^{1,\dots,1}]) = [1_R] - [1_R] = 0 \in H_0(\mathbb{\Delta}^{1,\dots,1}).$$

By Lemma 3.5, $[\Psi^{1,\dots,1}] = 0$, and the proof is complete. □

Fix a choice of $\psi^{\vec{\boldsymbol{d}}} \in \mathsf{Cobar}^{\Pi}(C(\mathbb{\Delta}^{\vec{\boldsymbol{d}}}))(v_0, v_d)$ provided by Lemma 3.6. For any topological space X, integer $k \ge 1$ and any two-indexed family of simplices

$$\sigma_{i,j} \colon \Delta^{d_{i,j}} \to X, \quad 1 \le i \le k, \quad 1 \le j \le n_i,$$

such that the image of the last vertex of each simplex coincides with the image of the first vertex of the next one, we denote by

$$\vec{\boldsymbol{\sigma}} = \bigvee_{i=1}^{k} \bigvee_{j=1}^{n_i} \sigma_{i,j} \colon \Delta^{\vec{\boldsymbol{d}}} \to X \tag{3.7}$$

the resulting map, which we call a *k-necklace in X*. Then, we define

$$\mathcal{H}^{d}_{X,k}(\otimes_{i=1}^{k} \boxtimes_{j=1}^{n_i} s^{-1}\sigma_{i,j}) = \vec{\boldsymbol{\sigma}}_*(\psi^{\vec{\boldsymbol{d}}}).$$

Clearly $\mathcal{H}^{d}_{X}$ is natural in X, and by a routine computation using the inductive hypothesis, equation (3.6) holds. This completes the proof of Theorem 1.4. □

3.3. **Elaboration on Remark 1.5.** For any topological space X with a basepoint $b \in X$, recall that $C^1(X,b) \subset C(X)$ denotes the dg subcoalgebra generated by 1-reduced simplices at b, i.e., simplices whose edges all map to b. Define the categorical subcoalgebra $\mathcal{C}^1(X,b) \subset \mathcal{C}(X)$ by setting $\mathcal{C}^1(X,b) = C^1(X,b)$ as R-modules and $\mathcal{S}(\mathcal{C}^1(X,b)) = \{b\}$. It is straightforward to check that $\mathcal{C}^1(X,b)$ is a categorical coalgebra with strict differential and zero curvature, hence a usual dg coalgebra, and that the linear isomorphism

$$\mathcal{C}^1(X,b) \cong C^1(X,b)$$

is an isomorphism of coalgebras in the usual sense.

Let us now examine the restriction of the A_∞-functor $\mathcal{F}_X$ and the functor $\mathcal{G}_X$ to the dg subcategory $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b))$ of $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))$. We denote these restrictions by $\mathcal{F}_X|_{\mathcal{C}^1}$ and $\mathcal{G}_X|_{\mathcal{C}^1}$, respectively.

First, we claim that if $u_n \in C_n(\mathbb{A}^n)$ is chosen as $u_n = \mathrm{id}_{\mathbb{A}^n}$ for all $n \geq 0$ in the construction of $\mathcal{I}_1$, then

$$\mathcal{F}_X|_{\mathcal{C}^1} \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b)) \to \mathsf{Cobar}^{\Pi}(C(X))$$

factors through the inclusion of dg categories

$$\mathsf{Cobar}^{\Pi}(C^1(X,b)) \subset \mathsf{Cobar}^{\Pi}(C(X)).$$

It suffices to check that for any $k \geq 1$, 1-reduced simplices $\sigma_m \colon \mathbb{A}^{d_m} \to X$ $(1 \leq m \leq k)$, $n \geq 0$, $1 \leq j \leq n$ and $1 \leq i \leq N(k-1,n)$, the restriction of the map

$$\begin{aligned} I^{d_1-1} \times \cdots \times I^{d_k-1} \times I^{k-1+n} &\to X \\ (x_1,\ldots,x_k,y) &\mapsto \\ &\mathrm{ev}_{n,j}\big(A_X(\sigma_1)(x_1) * \cdots * A_X(\sigma_k)(x_k), \tau^{k-1}_{n,i}(y, T_1(x_1),\ldots,T_k(x_k))\big) \end{aligned}$$

to any edge of the domain cube is the constant map to b. Here,

$$\mathrm{ev}_{n,j} \colon \mathsf{P}^n X(b,b) \to X$$

is the evaluation map at the j-th marked point, and $\tau^{k-1}_{n,i}$ is constructed in Lemma 3.2.

This can be checked as follows. There are two types of edges:

- A vertex in $I^{d_1-1} \times \cdots \times I^{d_k-1}$ with an edge in I^{k-1+n}. By the definition of Adams' map, such an edge corresponds to the concatenation of some edges of the 1-reduced simplices $\sigma_1,\ldots,\sigma_k$, hence is mapped to b.
- An edge in $I^{d_1-1} \times \cdots \times I^{d_k-1}$ with a vertex in I^{k-1+n}. By the choice of u_n and the explicit inductive construction of ξ^{k-1}_n in Lemma 3.2, the point $\tau^{k-1}_{n,i}(y,T_1,\ldots,T_k) \in \mathbb{A}^n_{T_1+\cdots+T_k}$ is a vertex of the scaled simplex $\mathbb{A}^n_{T_1+\cdots+T_k}$ whenever $y \in I^{k-1+n}$ is a vertex. Therefore such an edge is mapped to one of the endpoints of $A_X(\sigma_1)(x_1) * \cdots * A_X(\sigma_k)(x_k)$, namely the basepoint b.

Hence the claim follows.

Next, observe that the map (2.2) vanishes on $C^1(X,b)$, so

$$\mathcal{G}_X|_{\mathcal{C}^1} \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b)) \to \mathsf{Cobar}^{\Pi}(C(X))$$

is simply given by the inclusion induced from

$$\overline{\mathcal{C}^1(X,b)} \cong \overline{C^1(X,b)} \hookrightarrow C^1(X,b) \hookrightarrow C(X)$$

and the inclusion of a direct sum into a direct product. Moreover, since $s^{-1}\overline{C^1(X,b)}$ is concentrated in positive degrees,

$$\bigoplus_{n=0}^{\infty} R_b \otimes (s^{-1}\overline{C^1(X,b)})^{\otimes n} \otimes R_b = \prod_{n=0}^{\infty} R_b \otimes (s^{-1}\overline{C^1(X,b)})^{\otimes n} \otimes R_b.$$

Hence $\mathcal{G}_X$ identifies $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b)) = \mathsf{Cobar}(C^1(X,b))$ with the conormalization of $\mathsf{Cobar}^{\prod}(C^1(X,b))$, viewed as a subcategory of $\mathsf{Cobar}^{\prod}(C(X))$.

Now assume X is simply connected. By the results in Section 4, the inclusion

$$\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b)) \hookrightarrow \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))$$

is a quasi-equivalence. By comparing spectral sequences, the inclusion

$$\mathsf{Cobar}^{\prod}(C^1(X,b)) \hookrightarrow \mathsf{Cobar}^{\prod}(C(X))$$

is also a quasi-equivalence. The discussion above is summarized as follows.

Proposition 3.7. For any simply connected topological space X, the A_∞-functor

$$\mathcal{I}_X \colon \mathsf{C}^{\square}(\mathsf{P}X) \to \mathsf{Cobar}^{\prod}(C(X))$$

acts as a left A_∞-homotopy inverse of the functor

$$\mathcal{A}_X \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X)) \to \mathsf{C}^{\square}(\mathsf{P}X)$$

on the common subcategory $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b))$. More precisely, the inclusions of this subcategory into $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))$ and $\mathsf{Cobar}^{\prod}(C(X))$ are both quasi-equivalences, and the restriction of the composition $\mathcal{I}_X \circ \mathcal{A}_X$ to this subcategory is A_∞-homotopic to the conormalization inclusion $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b)) \hookrightarrow \mathsf{Cobar}^{\prod}(C^1(X,b))$.

Remark 3.8. One could alternatively *define* $\mathsf{Cobar}^{\prod}(C)$ by additionally modding out the ideal generated by C_0. On $\mathsf{Cobar}^{\prod}(C^1(X,b))$, this quotient is naturally isomorphic to the conormalization, so in the alternative definition, the map $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^1(X,b)) \hookrightarrow \mathsf{Cobar}^{\prod}(C^1(X,b))$ is the identity.

4. Adams' map and the universal cover

In this section, we give an elementary proof of Theorem 1.8 and Corollary 1.9 by passing to the universal cover and using Adams' classical cobar theorem for simply connected spaces. This section is essentially self-contained and does not use any of the results proved earlier. Corollary 1.9 was originally proved in [RZ18] using different methods.

Let X be a topological space. We may assume X is path-connected; otherwise we argue separately on each path-connected component of X. Fix once and for all, for each pair $(a,b) \in X \times X$ with $a \neq b$, a unit-length path

$$\gamma_{ab} \colon [0,1] \to X, \qquad \gamma_{ab}(0) = a,\ \gamma_{ab}(1) = b,$$

and require that $\gamma_{ba}(t) = \gamma_{ab}(1-t)$ for all $a, b \in X$ with $a \neq b$ and $t \in [0,1]$. For each $b \in X$, we also define a zero-length path

$$\gamma_{bb} \colon [0,0] \to X, \qquad \gamma_{bb}(0) = b.$$

Denote the resulting family of paths by

$$\mathcal{O}_X = \{\gamma_{ab}\}_{a,b\in X}. \tag{4.1}$$

For any two $a, b \in X$, there is a homotopy equivalence

$$L_{\gamma_{ab}} \colon \mathsf{P}X(b,b) \to \mathsf{P}X(a,b), \qquad \gamma \mapsto \gamma_{ab} * \gamma,$$

with homotopy inverse $L_{\gamma_{ba}} : \gamma' \mapsto \gamma_{ba} * \gamma'$. Similarly, for $a \neq b$, define an R-linear map

$$L^{\boxtimes}_{\gamma_{ab}} \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b) \to \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a,b), \qquad x \mapsto (s^{-1}\gamma_{ab}) \boxtimes x,$$

where γ_{ab} is regarded as a 1-simplex in X. Then $L^{\boxtimes}_{\gamma_{ab}}$ is a chain-homotopy equivalence with chain-homotopy inverse $L^{\boxtimes}_{\gamma_{ba}} : x' \mapsto (s^{-1}\gamma_{ba}) \boxtimes x'$, and a chain homotopy from the identity to $L^{\boxtimes}_{\gamma_{ba}} \circ L^{\boxtimes}_{\gamma_{ab}}$ is given by $x \mapsto (s^{-1}\sigma) \boxtimes x$, where $\sigma = [v_0, v_1, v_2]$ has edges $[v_0, v_1] = \gamma_{ba}$, $[v_1, v_2] = \gamma_{ab}$, and $[v_0, v_2]$ is the degenerate edge given by the constant path at b. We also include the case $a = b$ by setting $L^{\boxtimes}_{\gamma_{bb}} = \mathrm{id}$.

Clearly the maps $L_{\gamma_{ab}}$ and $L^{\boxtimes}_{\gamma_{ab}}$ are compatible with Adams' map. More precisely, there is a commutative diagram

$$\begin{array}{ccc} \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b) & \xrightarrow{\mathcal{A}_X} & C^{\square}(\mathsf{P}X)(b,b) \\ \simeq\Big\downarrow L^{\boxtimes}_{\gamma_{ab}} & & \simeq\Big\downarrow (L_{\gamma_{ab}})_* \\ \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a,b) & \xrightarrow{\mathcal{A}_X} & C^{\square}(\mathsf{P}X)(a,b). \end{array} \tag{4.2}$$

The diagram (4.2) implies that Theorem 1.8 is equivalent to the following proposition, whose proof is postponed to the end of this section.

Proposition 4.1. For any pointed path-connected space (X, b),

$$\mathcal{A}_X \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b) \to C^{\square}(\mathsf{P}X)(b,b)$$

is a quasi-isomorphism.

For any $b \in X$, consider the dg subcoalgebra $C^0(X,b)$ of $C(X)$ generated by simplices with all vertices at b. Let $\mathcal{C}^0(X,b)$ be the categorical subcoalgebra of $\mathcal{C}(X)$ with $\mathcal{C}^0(X,b) = C^0(X,b)$ as R-modules and $\mathcal{S}(\mathcal{C}^0(X,b)) = \{b\}$.

Define an R-linear map

$$\begin{aligned} E_- \colon s^{-1}\overline{C^0(X,b)} &\longrightarrow \big(R_b \boxtimes s^{-1}\overline{\mathcal{C}^0(X,b)} \boxtimes R_b\big) \oplus R_b, \\ s^{-1}\sigma &\longmapsto s^{-1}\sigma - e(\sigma), \qquad \sigma \colon (\Delta^n, \text{vertices}) \to (X, b), \end{aligned}$$

where e is (2.2). Extend E_- multiplicatively over $\boxtimes$. This induces a linear map

$$E_- \colon \mathsf{Cobar}\big(C^0(X,b)\big) \longrightarrow \mathsf{Cobar}^{\boxtimes}\big(\mathcal{C}^0(X,b)\big),$$

which is a chain map by straightforward calculation. Moreover, E_- is a dg algebra isomorphism, whose inverse E_+ is given by $s^{-1}\sigma \mapsto s^{-1}\sigma + e(\sigma)$ on generators.

Denote by

$$\iota_b \colon \mathcal{C}^0(X,b) \hookrightarrow \mathcal{C}(X)$$

the natural inclusion. Using the family $\mathcal{O}_X$ (4.1), one can deform each simplex in X to one with all vertices at b, proceeding by induction on the dimension of the simplex. This procedure induces a morphism of categorical coalgebras

$$f_{\mathcal{O}_X} \colon \mathcal{C}(X) \to \mathcal{C}^0(X,b)$$

satisfying $f_{\mathcal{O}_X} \circ \iota_b = \mathrm{id}_{\mathcal{C}^0(X,b)}$, as well as an R-linear map $H_{\mathcal{O}_X} : \mathcal{C}_\bullet(X) \to \mathcal{C}_{\bullet+1}(X)$ serving as a homotopy between $\mathrm{id}_{\mathcal{C}(X)}$ and the morphism $\iota_b \circ f_{\mathcal{O}_X}$. Consequently,

$\mathsf{Cobar}^{\boxtimes}(f_{\mathcal{O}_X}) \circ \mathsf{Cobar}^{\boxtimes}(\iota_b)$ is the identity on $\mathsf{Cobar}^{\boxtimes}(\mathcal{C}^0(X,b))(b,b)$, and the extension of $H_{\mathcal{O}_X}$ to a derivation, $\widehat{H}_{\mathcal{O}_X}\colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b) \to \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b)$, is a chain homotopy between the identity and $\mathsf{Cobar}^{\boxtimes}(\iota_b) \circ \mathsf{Cobar}^{\boxtimes}(f_{\mathcal{O}_X})$. Therefore, the natural inclusion

$$\mathsf{Cobar}^{\boxtimes}(\iota_b)\colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}^0(X,b)) \hookrightarrow \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b)$$

is a chain-homotopy equivalence.

Clearly the maps E_- and $\mathsf{Cobar}^{\boxtimes}(\iota_b)$ are compatible with Adams' map, i.e., there is a commutative diagram

(4.3)

$$\begin{array}{ccccc} \mathsf{Cobar}(C^0(X,b)) & \xrightarrow[\cong]{E_-} & \mathsf{Cobar}^{\boxtimes}(\mathcal{C}^0(X,b))(b,b) & \xhookrightarrow[\simeq]{\mathsf{Cobar}^{\boxtimes}(\iota_b)} & \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b) \\ & \searrow^{\mathcal{A}_X} & \downarrow \mathcal{A}_X & \swarrow^{\mathcal{A}_X} & \\ & & C^{\square}(\mathsf{P}X)(b,b). & & \end{array}$$

The diagram (4.3) implies that Corollary 1.9 is equivalent to Proposition 4.1.

Remark 4.2. Adams' original proof [Ada56] shows that

$$\mathcal{A}_X\colon \mathsf{Cobar}(C^1(X,b)) \longrightarrow C^{\square}(\mathsf{P}X)(b,b)$$

is a quasi-isomorphism if X is simply connected. Note that the natural inclusion

$$\mathsf{Cobar}(C^1(X,b)) \hookrightarrow \mathsf{Cobar}(C^0(X,b))$$

is compatible with Adams' map, and is a quasi-isomorphism provided that X is simply connected. Consequently, by (4.2)(4.3),

$$\mathcal{A}_X\colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a,b) \longrightarrow C^{\square}(\mathsf{P}X)(a,b)$$

is a quasi-isomorphism for all $a,b \in X$ whenever X is simply connected.

Proof of Proposition 4.1. For any $b \in X$, denote by $\widetilde{X}_b$ the universal cover of X at b and denote by

$$\pi_b\colon (\widetilde{X}_b, [c_b]) \to (X,b)$$

be the universal covering map, where $[c_b]$ denotes the homotopy class of the constant path $c_b\colon [0,1] \to X$ at b.

First, by lifting paths in X to $\widetilde{X}_b$, we obtain a homeomorphism of spaces

$$\mathsf{P}X(b,b) \cong \bigsqcup_{\alpha \in \pi_1(X,b)} \mathsf{P}\widetilde{X}_b([c_b],\alpha),$$

which induces an isomorphism of chain complexes

$$\mathsf{C}^{\square}(\mathsf{P}X)(b,b) \cong C^{\square}\Big(\bigsqcup_{\alpha\in\pi_1(X,b)} \mathsf{P}\widetilde{X}_b([c_b],\alpha)\Big) = \bigoplus_{\alpha\in\pi_1(X,b)} \mathsf{C}^{\square}(\mathsf{P}\widetilde{X}_b)([c_b],\alpha). \tag{4.4}$$

Next, by lifting necklaces in X to $\widetilde{X}_b$, we obtain a chain complex isomorphism

$$\mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(b,b) \cong \bigoplus_{\alpha\in\pi_1(X,b)} \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(\widetilde{X}_b))([c_b],\alpha). \tag{4.5}$$

More precisely, recall that $\mathbb{A}^{d_1} \vee \cdots \vee \mathbb{A}^{d_k}$ is the wedge sum of $\mathbb{A}^{d_1}, \ldots, \mathbb{A}^{d_k}$ formed by successively identifying the last vertex of each simplex with the first vertex of the next. Every necklace in X

$$\boldsymbol{\sigma}\colon (\mathbb{A}^{d_1} \vee \cdots \vee \mathbb{A}^{d_k}, v_0, v_d) \to (X,b,b)$$

uniquely lifts to a necklace in $\widetilde{X}_b$

$$\widetilde{\boldsymbol{\sigma}}\colon (\mathbb{A}^{d_1} \vee \cdots \vee \mathbb{A}^{d_k}, v_0, v_d) \to (\widetilde{X}_b, [c_b], \pi^{-1}(b))$$

such that $\pi_b \circ \widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$, where $d = d_1 + \cdots + d_k$ and v_i denotes the $(i+1)$-th vertex of $\mathbb{A}^{d_1} \vee \cdots \vee \mathbb{A}^{d_k}$. The endpoint $\widetilde{\boldsymbol{\sigma}}(v_d)$ is uniquely determined by the based homotopy class of $\boldsymbol{\sigma} \circ \gamma$ for an arbitrary path γ in $\mathbb{A}^{d_1} \vee \cdots \vee \mathbb{A}^{d_k}$ from v_0 to v_d. Conversely, every necklace in $\widetilde{X}_b$ arises as the lift of a unique necklace in X. Thus, for each $k \geq 0$ there is an R-module isomorphism

$$R_b \boxtimes (s^{-1}\overline{\mathcal{C}(X)})^{\boxtimes k} \boxtimes R_b \cong \bigoplus_{\alpha \in \pi_1(X,b)} R_{[c_b]} \boxtimes (s^{-1}\overline{\mathcal{C}(\widetilde{X}_b)})^{\boxtimes k} \boxtimes R_\alpha,$$

and taken together, these induce the isomorphism (4.5).

By Remark 4.2, for any $\alpha \in \pi_1(X, b)$,

$$\mathcal{A}_{\widetilde{X}_b}\colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(\widetilde{X}_b))([c_b], \alpha) \to C^{\square}(\mathsf{P}\widetilde{X}_b)([c_b], \alpha)$$

is a quasi-isomorphism. The proof of the proposition is then completed by observing that the isomorphisms (4.4) and (4.5) are compatible with Adams' map. □

5. Extension of the results to arbitrary spaces

In this section we give the constructions of $\mathsf{C}^{\mathsf{P}}(X)$, $\widetilde{It}_X$, $\widetilde{\mathcal{I}}_X$, $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{G}}_X$, and $\widetilde{\mathcal{H}}_X$ for any topological space X, thereby proving Theorem 1.10. The constructions of $\mathsf{C}^{\mathsf{P}}(X)$ and $\widetilde{It}_X$ are the key ingredients; once these are established, the remaining results follow by arguments parallel to those used for Theorem 1.2 and Theorem 1.4.

We define the dg category $\mathsf{C}^{\mathsf{P}}(X)$ as follows. The objects are the points of X. For any $a, b \in X$, the morphism complex is

$$\mathsf{C}^{\mathsf{P}}(X)(a,b) = \bigoplus_{\beta \in \Pi_1 X(a,b)} \mathsf{Cobar}^{\Pi}(C(\widetilde{X}_a))([c_a], \beta),$$

where $\Pi_1 X$ denotes the fundamental groupoid of X, $(\widetilde{X}_a, [c_a])$ is the universal covering space of (X, a), and $\beta \in \Pi_1 X(a,b)$ is viewed as a point of $\widetilde{X}_a$ via the canonical inclusion

$$\Pi_1 X(a,b) \hookrightarrow \bigcup_{c \in X} \Pi_1 X(a,c) = \widetilde{X}_a.$$

The composition of morphisms is induced by the map

$$\begin{gathered}(R_{[c_a]} \otimes (s^{-1}C(\widetilde{X}_a))^{\otimes n} \otimes R_\beta) \otimes (R_{[c_b]} \otimes (s^{-1}C(\widetilde{X}_b))^{\otimes n'} \otimes R_\gamma) \\ \to R_{[c_a]} \otimes (s^{-1}C(\widetilde{X}_a))^{\otimes n+n'} \otimes R_{\beta\circ\gamma} \\ (1_R \otimes x \otimes 1_R) \otimes (1_R \otimes y \otimes 1_R) \mapsto 1_R \otimes (x \otimes \beta_*(y)) \otimes 1_R\end{gathered}$$

for any $a, b, c \in X$, $\beta \in \Pi_1 X(a,b)$ and $\gamma \in \Pi_1 X(b,c)$, where β_* is the map on the tensor factors of singular chains induced by the homeomorphism $\widetilde{X}_b \to \widetilde{X}_a$, $z \mapsto \beta z$.

For any $a, b \in X$, define $\widetilde{It}_X$ to be the comoposition of chain maps

$$\mathsf{C}^{\square}(\mathsf{P}X)(a,b) \cong \bigoplus_{\beta \in \Pi_1 X(a,b)} \mathsf{C}^{\square}(\mathsf{P}\widetilde{X}_a)([c_a], \beta) \to \mathsf{C}^{\mathsf{P}}(X)(a,b), \tag{5.1}$$

where the first map is defined in the same manner as (4.4), and the second map is induced by $It_{\widetilde{X}_a}$ in (2.3) on each direct summand.

Remark 5.1. The construction of $\mathsf{C}^{\mathsf{P}}(X)$ and It_X is motivated by an idea of Irie. Irie proposed a chain model for the free loop space (path space) and a version of the iterated integral map based on "de Rham chains" as a simplification of his construction in [Iri17], which was established by the second-named author in [Wan23, Chapter 2]. This approach, unlike C^{P}, uses the fundamental groupoid without referring to universal covering spaces.

We extend $\widetilde{It}_X$ to an A_∞-functor

$$\widetilde{\mathcal{I}}_X = \{\widetilde{\mathcal{I}}_{X,k}\}_{k\geq 1} \colon \mathsf{C}^{\square}(\mathsf{P}X) \to \mathsf{C}^{\mathsf{P}}(X)$$

with $\widetilde{\mathcal{I}}_{X,1} = \widetilde{It}_X$ as follows. Recall the cosimplicial space $[n] \mapsto \mathsf{P}^n X(a,b)$ in (3.2) for $a, b \in X$. There is a sequence of canonical isomorphisms of chain complexes

$$C(\mathsf{P}^n X(a,b)) \cong \bigoplus_{\beta\in\Pi_1 X(a,b)} C(\mathsf{P}^n \widetilde{X}_a([c_a],\beta)), \qquad n \geq 0$$

which is compatible with the cosimplicial structures, and induces an inclusion

$$\bigoplus_{\beta\in\Pi_1 X(a,b)} \mathrm{Tot}^{\Pi}\big(\{C(\mathsf{P}^n \widetilde{X}_a([c_a],\beta))\}_{n\geq 0}\big) \hookrightarrow \mathrm{Tot}^{\Pi}\big(\{C(\mathsf{P}^n X(a,b))\}_{n\geq 0}\big).$$

For every $k \geq 1$, the linear map $\mathcal{J}_{X,k}$ from Lemma 3.1 factors through the above inclusion with $a_0 = a$, $a_k = b$. Then we define

$$\widetilde{\mathcal{I}}_{X,k} \coloneqq \widetilde{AW} \circ \widetilde{ev}_* \circ \mathcal{J}_{X,k} \colon \bigotimes_{i=1}^{k} \mathsf{C}^{\square}(\mathsf{P}X)(a_{i-1},a_i) \to \mathsf{C}^{\mathsf{P}}(X)(a_0,a_k),$$

where the maps $\widetilde{\mathrm{ev}}_*$, $\widetilde{AW}$ are induced, on each direct summand, by

$$\mathrm{ev}_* \colon \mathrm{Tot}^{\Pi}\big(\{C(\mathsf{P}^n \widetilde{X}_a([c_a],\beta))\}_{n\geq 0}\big) \to \mathrm{Tot}^{\Pi}\big(\{C(\{[c_a]\} \times (\widetilde{X}_a)^n \times \{\beta\})\}_{n\geq 0}\big)$$

$$\text{and } AW \colon \mathrm{Tot}^{\Pi}\big(\{C(\{[c_a]\} \times (\widetilde{X}_a)^n \times \{\beta\})\}_{n\geq 0}\big) \to \mathsf{Cobar}^{\Pi}(C(\widetilde{X}_a))([c_a],\beta)$$

as appearing in (3.3) and (3.1), respectively.

We define the functor

$$\widetilde{\mathcal{G}}_X \colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X)) \to \mathsf{C}^{\mathsf{P}}(X)$$

as follows. It acts as the identity on objects, and for $a, b \in X$ its action on morphisms is given by the composition of chain maps

$$\mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X))(a,b) \cong \bigoplus_{\beta\in\Pi_1 X(a,b)} \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(\widetilde{X}_a))([c_a],\beta) \to \mathsf{C}^{\mathsf{P}}(X),$$

where the first map is defined in the same manner as (4.5), and the second map is induced by $\mathcal{G}_{\widetilde{X}_a}$ in (2.8) on each direct summand.

We construct an A_∞-homotopy $\widetilde{\mathcal{H}}_X$ between $\widetilde{\mathcal{F}}_X = \widetilde{\mathcal{I}}_X \circ \mathcal{A}_X$ and $\widetilde{\mathcal{G}}_X$ as follows. For any $k \geq 1$ and any k-necklace in X

$$\vec{\boldsymbol{\sigma}} \colon \mathbb{A}^{\vec{\boldsymbol{d}}} \to X, \qquad \text{where } \vec{\boldsymbol{d}} = (\boldsymbol{d}_1,\ldots,\boldsymbol{d}_k),\ \boldsymbol{d}_i = (d_{i,1},\ldots,d_{i,n_i}),$$

as in (3.7), $\vec{\boldsymbol{\sigma}}$ uniquely lifts to a k-necklace in $\widetilde{X}_{\vec{\boldsymbol{\sigma}}(v_0)}$

$$\widetilde{\vec{\boldsymbol{\sigma}}} \colon (\mathbb{A}^{\vec{\boldsymbol{d}}}, v_0) \to (\widetilde{X}_{\vec{\boldsymbol{\sigma}}(v_0)}, [c_{\vec{\boldsymbol{\sigma}}(v_0)}])$$

such that $\pi_{\vec{\boldsymbol{\sigma}}(v_0)} \circ \widetilde{\vec{\boldsymbol{\sigma}}} = \vec{\boldsymbol{\sigma}}$, where v_0 denotes the first vertex of $\mathbb{A}^{\vec{\boldsymbol{d}}}$. Then we define

$$\widetilde{\mathcal{H}}_X(\otimes_{i=1}^{k} \boxtimes_{j=1}^{n_i} s^{-1}\sigma_{i,j}) = \widetilde{\vec{\boldsymbol{\sigma}}}_*(\psi^{\vec{\boldsymbol{d}}})$$

where $\psi^{\vec{d}} \in \mathsf{Cobar}^{\prod}(C(\mathbb{\Delta}^{\vec{d}}))(v_0, v_d)$ is provided in Lemma 3.6.

By construction, $\widetilde{\mathcal{I}}_X$, $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{G}}_X$, and $\widetilde{\mathcal{H}}_X$ are natural in X. Theorem 1.10 follows.

Remark 5.2. There is a natural transformation

$$\pi\colon \mathsf{C}^{\mathsf{P}} \Rightarrow \mathsf{Cobar}^{\prod} \circ C^{\mathbb{\Delta}}$$

such that for any topological space X, π_X is the identity on objects, and

$$\pi_X\colon \mathsf{C}^{\mathsf{P}}(X)(a,b) \to \mathsf{Cobar}^{\prod}(C(X))(a,b), \qquad a,b \in X$$

is induced by the universal covering map $\pi_a\colon (\widetilde{X}_a, [c_a]) \to (X, a)$ on each direct summand. It is straightforward to check that $\mathcal{I}_X$, $\mathcal{F}_X$, $\mathcal{G}_X$ and $\mathcal{H}_X$ are the compositions of π_X with $\widetilde{\mathcal{I}}_X$, $\widetilde{\mathcal{F}}_X$, $\widetilde{\mathcal{G}}_X$ and $\widetilde{\mathcal{H}}_X$, respectively. If X is simply connected, then π_X is an isomorphism of dg categories.

It seems hard to formulate an analogue of Remark 1.5 in the context here. Nonetheless, the following is true.

Proposition 5.3. For any topological space X, the functor $\widetilde{\mathcal{G}}_X$ is a quasi-equivalence.

Proof. We know from Section 3.3 that $\mathcal{G}_X$ is a quasi-equivalence for simply connected X. The lemma follows by applying this to $\widetilde{X}_a$ in the construction of $\widetilde{\mathcal{G}}_X$. □

Theorem 1.10 and Proposition 5.3 imply that:

Corollary 5.4. For any topological space X:

(1) The functor

$$\mathcal{A}_X\colon \mathsf{Cobar}^{\boxtimes}(\mathcal{C}(X)) \to \mathsf{C}^{\square}(\mathsf{P}X)$$

induced by Adams' map is a quasi-equivalence if and only if the A_∞-functor

$$\widetilde{\mathcal{I}}_X\colon \mathsf{C}^{\square}(\mathsf{P}X) \to \mathsf{C}^{\mathsf{P}}(X)$$

motivated by Chen's iterated integral map is an A_∞-quasi-equivalence.

(2) (Given Theorem 1.8) The A_∞-functor $\widetilde{\mathcal{I}}_X$ is an A_∞-quasi-equivalence.

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Manuel Rivera, Department of Mathematics, Purdue University, 150 N University St, West Lafayette, Indiana, 47907

Email address: `manuelr@purdue.edu`

Yi Wang, Department of Mathematics, Purdue University, 150 N University St, West Lafayette, Indiana, 47907

Email address: `wang6206@purdue.edu`