# UNDECIDABILITY OF THE QUASI-ISOMORPHISM PROBLEM FOR SEMI-FREE DG ALGEBRAS VIA THE COBAR CONSTRUCTION

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In [3], proofs to Theorem 2 are presented based on autonomous responses of *Aletheia*, an AI math research agent. Below, I explain how Theorem 2 is a straightforward consequence of the main result of [4] and the undecidability of the triviality problem for finitely presented groups. I believe the argument is conceptually clarifying. In fact, given any finitely presented group, the cobar construction algorithmically provides a semi-free finitely presented differential graded (dg) algebra whose zeroth Hochschild homology is the free module on the set of conjugacy classes of the group. The *Aletheia* raw output linked in [3] contains an attempt to a solution using this result, which the authors do not discuss. We first fix notation and recall the main theorem of [4]. Fix a (non-trivial) commutative ring with unit $R$.

(1) Denote by $\Omega\colon \mathsf{dgCoalg}_R \to \mathsf{dgAlg}_R$ the *cobar* functor from the category of dg coassociative coaugmented $R$-coalgebras to the category of dg associative augmented $R$-algebras. For any $C \in \mathsf{dgCoalg}_R$, the dg algebra $\Omega(C)$ is *semi-free*, its underlying graded algebra being the free associative graded algebra on $s^{-1}\overline{C}$, where $\overline{C}$ is the cokernel of the coaugmentation and $s^{-1}$ the "shift down by 1" functor. The differential is defined by extending the differential and coproduct of $C$ by the graded Leibniz rule.

(2) For any simplicial set $X$, denote by $C_*(X;R)$ the dg coassociative $R$-coalgebra of normalized chains on $X$ with Alexander-Whitney coproduct. A choice of vertex in $X$ gives a coaugmentation $R \to C_*(X;R)$. For any topological space $Y$, denote by $S_*(Y;R)$ the dg $R$-module of normalized singular chains on $Y$. A topological monoid structure on $Y$ induces a dg associative algebra structure on $S_*(Y,R)$.

(3) If $X$ is a simplicial set with a single vertex and $\sigma \in X_{n>0}$ a non-degenerate $n$-simplex, the differential $d\colon \Omega(C_*(X;R)) \to \Omega(C_*(X;R))$ is given explicitly by

$$d(s^{-1}\sigma) = \sum_{i=0}^{n}(-1)^{i+1}s^{-1}\sigma(0,...,\hat{i},...,n) + \sum_{j=1}^{n-1}(-1)^{j}s^{-1}\sigma(0,...,j)\otimes s^{-1}\sigma(j,...,n),$$

where $\sigma(i_0,...,i_k)$ denotes the $k$-simplex obtained by restricting $\sigma$ to the vertices $i_0,...,i_k$.

(4) Denote by $\Omega\colon \mathsf{Top}^* \to \mathsf{Mon}_{\mathsf{Top}}$ the *based (Moore) loops* functor from the category of pointed topological spaces to the category of topological monoids.

(5) The *homotopy category* of a simplicial set $X$ is the category $\pi(X)$ obtained by applying the left adjoint of the nerve functor.

(6) Two dg associative algebras are *quasi-isomorphic* if they are connected by a zig-zag of maps of dg associative algebras each inducing an isomorphism on homology.

**Theorem 1** ([4])**.** *If $X$ is a simplicial set with a single vertex and $\pi(X)$ is a group, then the dg associative algebras $\Omega C_*(X;R)$ and $S_*(\Omega|X|;R)$ are naturally quasi-isomorphic.*

We now deduce the main theorem.

**Theorem 2.** *If $R$ is Turing computable, the problem of determining whether two semi-free finitely presentable differential graded associative $R$-algebras are quasi-isomorphic is undecidable.*

*Proof of Theorem 2.* We reduce the statement to the triviality problem for finitely presented groups, which is known to be undecidable. Given any finitely presented group $(G,P)$ we associate two explicit semi-free, finitely presented, dg algebras $A_P$ and $B_P$ which are quasi-isomorphic if and only if $G$ is trivial. First, one can effectively associate to $(G,P)$ a 2-skeletal simplicial set $K'_P$ with a single vertex and a finite number of simplices modeling the presentation complex of $P$. Glue a copy of $\Delta^3/\sim$, where

$\sim$ collapses $[0,2]$ to a vertex $[0]=[2]$ and $[1,3]$ to $[1]=[3]$, along the 1-simplex $[1,2]$ to each 1-simplex of $K'_P$. Denote by $K_P$ the resulting simplicial set. Then $\pi(K_P) \cong \pi_1(|K_P|) \cong \pi_1(|K'_P|) \cong G$. Define $A_P = \Omega C_*(K_P;R)$. By Theorem 1, $A_P$ is quasi-isomorphic to $A'_P = S_*(\Omega|K_P|;R)$. If $G$ is the trivial group, then $A_P$ is quasi-isomorphic to $B_P = \Omega C_*(\bigvee_{i=1}^{k(P)} S^2;R)$, where $k(P)$ the number of relations in $P$ and $S^2$ is the simplicial set $\Delta^2/\partial\Delta^2$. Conversely, if $A_P$ and $B_P$ are quasi-isomorphic, by Theorem 1 and the quasi-isomorphism invariance of Hochschild homology, we have isomorphisms

$$HH_*(A'_P, A'_P) \cong HH_*(A_P, A_P) \cong HH_*(B_P, B_P) \cong HH_*(B'_P, B'_P),$$

where $B'_P = S_*(\Omega|\bigvee_{i=1}^{k(P)} S^2|;R)$. By section V.1 of [2], there are isomorphisms

$$H_*(L|K_P|;R) \cong HH_*(A'_P, A'_P) \cong HH_*(B'_P, B'_P) \cong H_*(L|\bigvee_{i=1}^{k(P)} S^2|;R)$$

where $L$ denotes the free loop space functor. Since the zeroth homology of the free loop space is the free $R$-module generated by the set of conjugacy classes of the fundamental group, it follows that $G = \pi_1(|K_P|)$ has a single conjugacy class, so $G$ is trivial. □

**Remarks.**

(1) If two dg algebras are quasi-isomorphic then they are derived Morita equivalent and Hochschild homology is derived Morita equivalence invariant. Hence, Theorem 2 holds when "quasi-isomorphic" is replaced by "derived Morita equivalent".

(2) The cobar functor $\Omega$ does not send quasi-isomorphisms of dg coalgebras to quasi-isomorphisms of dg algebras. If $X$ is a simplicial set with exactly one vertex, one non-degenerate 1-simplex, and every other simplex degenerate, the natural map $f\colon C_*(X;R) \to S_*(|X|;R)$ is a quasi-isomorphism of dg coaugmented coalgebras but $H_0(\Omega(f))$ is the inclusion $R[x] \to R[x, x^{-1}]$.

(3) If $C$ and $C'$ are *simply connected* dg coalgebras (i.e. $R = C_0 = C'_0$, $0 = C_1 = C'_1$, and $0 = C_i = C'_i$ for $i < 0$), any quasi-isomorphism $f\colon C \to C'$ induces a quasi-isomorphism $\Omega(f)\colon \Omega(C) \to \Omega(C')$. This follows from a standard spectral sequence argument.

(4) A quasi-isomorphism of *homologically* simply connected dg coaugmented coalgebras may not induce a quasi-isomorphism of dg algebras after applying $\Omega$. Let $Y$ be a pointed space with non-trivial perfect fundamental group and let $Y^+$ be its Quillen plus construction. Then $0 = H_1(Y;R) = H_1(Y^+;R)$ and there is a quasi-isomorphism $S_*(Y;R) \to S_*(Y^+;R)$, that does not induce a quasi-isomorphism of dg algebras after applying $\Omega$.

(5) Suppose $X$ and $X'$ are simplicial sets with a single vertex. If $f\colon X \to X'$ is a *categorical equivalence* (*Joyal equivalence*) then $\Omega C_*(f;R)\colon \Omega C_*(X;R) \to \Omega C_*(X';R)$ is a quasi-isomorphism of dg algebras. Any homotopy equivalence between simplicial sets whose homotopy categories are groupoids is a categorical equivalence.

(6) Two better questions are the following: *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are semi-free finitely presented dg associative algebras that are connected, i.e. $\mathcal{A}_0 = R = \mathcal{B}_0$, and non-negatively graded. Is the problem of determining whether $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic/derived Morita equivalent algorithmically decidable? What if $\mathcal{A}$ and $\mathcal{B}$ are required to be graded commutative?* See [1].

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## References

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